Existence and uniqueness of the Competitive Equilibrium for Infinite Dimensional Economies

Elvio Accinelli

Existence and Uniqueness of the Competitive Equilibrium for Infinite Dimensional Economies

Elvio Accinelli *

Abstract

In this paper, from the excess utility function we obtain a binary relation in the social weights space and then, for an infinite dimensional economy, we prove the existence of equilibrium, in our approach we don't suppose the existence of a demand function. Finally, we obtain a condition for the uniqueness of equilibrium, and we give some examples of economies that satisfy this condition.

Introduction

In this paper without assuming the existence of the demand function, we prove from the excess utility function an existence of equilibrium theorem, and we obtain a condition to uniqueness of equilibrium. The introduction of the excess utility function, allow us to transform an infinite dimensional problem in a finite dimensional case.

In the first section we characterize the model, and we introduce some standard definition in general equilibrium theory. In the second section we introduce the excess utility function and we show some of its properties. In the third section from the excess utility function we prove that there exists a bijective relation between the equilibrium allocations set and the set of zeros of excess utility function. In the fourth part from the excess utility function we obtain a binary relation in the social weights space, we prove that the equilibrium set is not empty. Our main tool is the Knaster, Kuratowski, Masurkiewicz lemma.

In the next section we define from the excess utility function the weak axiom of the revealed preference. So defined, this axiom, is only formally similar with the classic one. It has the same mathematical properties that the classic axiom of revealed preference but it has not the same economical interpretation. We prove that if the excess utility function has this property then uniqueness of equilibrium follows, that is there exists only one zero for this function. Finally

*I wish to thank Aloisio Araujo, Paulo K. Monteiro, Flavio Menezes, Daniel Vaz and Ricardo Marchesini for useful comments.

examples of economies with weak axiom of revealed preference in the excess utility function are giving.

1 The Model

Let us consider a pure exchange economy with n agents and l goods at each state of the world. The set of states is a measure space : $(\Omega, \mathcal{A}, \nu)$.

We assume that each agent has the same consumption space, $\mathcal{M} = \Pi_{j=1}^{l} \mathcal{M}_j$ where $\mathcal{M}_j$ is the space of all positive measurable functions defined on $(\Omega, \mathcal{A}, \nu)$.

Let be $R^l_{++} = \{x \in R^l$ with all components positives$\}$.

Following [MC] we consider the space Λ of the C^2 utility functions on R^l_{++}, strictly monotone, differentiably strictly concave and proper.

Definition 1 *A C^2 utility function u is differentiably strictly convex, if it is strictly convex and every point is regular; that is the gaussian curvature, C_x of each level surface of u, is a non null function in each x.*

For $x, y \in R^l$ we will write $x > y$ if $x_i \geq y_i$ $i = 1 \ldots l$ and $x \neq y$.

Definition 2 *A utility function is strictly monotone if $x > y \Rightarrow u(x) > u(y)$.*

Definition 3 *We say that $u \in C^2$ is proper if the limit of $|u'(x)|$ is infinite, when x approach to the boundary of R^l_{++}, i.e: the set $B = \{x : x_i = 0 \text{ for some } i = 1, \ldots, n\}$.*

We will consider the space U of all measurable functions $U : \Omega \times R^l_{++} \rightarrow R$, such that $U(s, \cdot) \in \Lambda$ for each $s \in \Omega$.

We introduce the uniform convergence in this space: $U_n \rightarrow U$ if $\|U_n - U\|_K \rightarrow 0$ for any compact $K \subset R^l_{++}$, where $\|U_n - U\|_K =$

$$ess \sup_{s\in\Omega} \max_{z \,\in\, K} \left\{ |U_n(s,z) - U(s,z)| + |\partial U_n(s,z) - \partial U(s,z)| + |\partial^2 U_n(s,z) - \partial^2 U(s,z)| \right\}.$$

Each agent is characterized by his utility function u_i and by his endowment $w_i \in \mathcal{M}$.

From now on we will work with economies with the following characteristics:

a) The utility functions $u_i : \mathcal{M} \rightarrow \mathcal{R}$ are separable. This means that they can be represented by
$$u_i(x) = \int_\Omega U_i(s, x(s)) d\nu(s) \quad i = 1, \ldots, n \tag{1}$$
where $U_i : \Omega \times R^l_{++} \rightarrow R$ and $U_i(s, \cdot)$ is for each agent his utility function at every state $s \in \Omega$.

b) The utility functions $U_i(s,.)$ belongs to a fixed compact subset of Λ, for each $s \in \Omega$ and $U_i \in \mathcal{U}$.

c) The agents' endowments, $w_i \in \mathcal{M}$ are bounded above and bounded away from zero in any component, i.e. there exists, h and ąl H with h $<\ w_{i_j}(s)\ <$ H for each $j = 1 \dots l,$ and $s \in \Omega$.

The following definitions are standard.

Definition 4 *An allocation of commodities is a list $x = (x_1, \dots, x_n)$ where $x : \Omega \to R^{ln}$ and $\sum_{k=1}^{n} x_k(s)\ \leq\ \sum_{k=1}^{n} w_k(s)$.*

Definition 5 *A commodity price system is a measurable function $p :\ \Omega\ \to R^l_{++}$, and for any $z \in R^l$ we denote by $\langle p, z\rangle$ the real number $\int_\Omega p(s)z(s)d\nu(s)$. (We are not using any specific symbol for the euclidean inner product in R^l.)*

Definition 6 *The pair (p, x) is an equilibrium if:*

i) p is a commodity price system and x is an al location,

ii) $\langle p, x_i\rangle\ \leq\ \langle p, w_i\rangle\ <\ \infty\ \ \forall\, i \in \{1, \dots, n\}$

iii) if $\langle p, z\rangle \leq \langle p, w_i\rangle$ with $\ z : \Omega \to R^l_{++}$, then

$$\int_\Omega U_i(s, x_i(s))d\nu(s) \geq \int_\Omega U_i(s, z(s))d\nu(s) \quad \forall\ \ i \in \{1, \dots, n\}.$$

2 The Excess Utility Function

In order to obtain our results we introduce the excess utility function.

We begin by writing the following well known proposition:

Proposition 1 *For each λ in the $(n-1)$ dimensional open simplex, $\Delta^{n-1} = \{\lambda \in R^n_{++}; \sum \lambda_i = 1\}$, there exists $\bar{x}(\lambda)\ =\ \{\bar{x}_1(\lambda), \cdots, \bar{x}_n(\lambda)\} \in R^{ln}_{++}$ solution of the following problem:*

$$\begin{array}{c} \max_{x \in R^{ln}} \sum_i \lambda_i U_i(x_i) \\ \textit{subject to}\ \ \sum_i x_i \leq \sum_i w_i\ \ \textit{and}\ \ x_i \geq 0. \end{array} \tag{2}$$

If U_i depend also on $s \in \Omega$, and $U_i(s, \cdot) \in \Lambda$ for each $s \in \Omega$, and $\lambda \in \triangle^{n-1}$, there exists $\bar{x}(s, \lambda) = \bar{x}_1(s, \lambda), ..., \bar{x}_n(s, \lambda)$ solution of the following problem:

$$\begin{array}{c} \max_{x(s) \in R^{ln}} \sum_i \lambda_i U_i(s, x_i(s)) \\ \text{subject to}\ \ \sum_i x_i(s) \leq \sum_i w_i(s)\ \ \text{and}\ \ x_i(s) \geq 0. \end{array} \tag{3}$$

If $\gamma^j(s,\lambda)$ are the Lagrange multipliers of the problem (3), $j \in \{1,\dots l\}$, then from the first order conditions we have that

$$\lambda_i \frac{\partial U_i(s,x(s,\lambda))}{\partial x^j} = \gamma^j(s,\lambda) \text{ with } i \in \{1,\dots,n\} \text{ and } j \in \{1,\dots,l\}$$

Then the following identities hold

$$\lambda_i \partial U_i(s,x(s,\lambda)) = \gamma(s,\lambda) \quad \forall\, i = 1,...,n\ ;\ \text{and}\ \forall\, s \in \Omega. \tag{4}$$

Remark 1 *From the Inada condition of "infinite marginal utility" at zero (Definition 3), the solution of (3) must be strictly positive almost everywhere. Since $U(s,.)$ is a monotone function, we can deduce that $\sum_{i=1}^n \bar{x}_i(s) = \sum_{i=1}^n w_i(s)$.*

Let us now define the excess utility function.

Definition 7 *Let $x_i(s,\lambda); i \in \{1,\dots,n\}$ be a solution of (3).*

We say that $e : \triangle^{n-1} \to R^n$ $e(\lambda) = (e_1(\lambda),...,e_n(\lambda))$, with

$$e_i(\lambda) = \frac{1}{\lambda_i}\int_\Omega \gamma(s,\lambda)[x_i(s,\lambda) - w_i(s)]d\nu(s),\ i = 1,\dots,n. \tag{5}$$

is the excess utility function.

Lemma 1 *The excess utility function is bounded for above, that is, there exists $k \in R$ such that $e(\lambda) \le k\mathbf{1}$, where $\mathbf{1}$ is a vector with all its components equal to 1.*

Proof: To prove this property, note that from definition we can write

$$e_i(\lambda) = \int_\Omega \partial U_i(s,x_i(\lambda))[x_i(s,\lambda) - w_i(s)]d\nu(s).$$

From the concavity of U_i it follows that:

$$U_i(s,x(s,\lambda)) - U_i(s,w(s)) \ge \partial U_i(s,x(s,\lambda))(x_i(s,\lambda) - w(s)).$$

Therefore,

$$e_i(\lambda) \le \int_\Omega U_i(s,x_i(s,\lambda)) - U_i(w_i(s))\, d\nu(s) \le \int_\Omega U_i(\sum_{j=1}^n w_j(s))\, d\nu(s),\ \forall\, \lambda.$$

If we let

$$k_i = \int_\Omega U_i(\sum_{i=1}^n w_i(s))\, d\nu(s) \ \text{ and } \quad k = \sup_{1\le i\le n} k_i$$

Property follows.

Remark 2 *Since the solution of (3) is homogeneous of degree zero: i.e, $\bar{x}(s,\lambda) = \bar{x}(s,\alpha\lambda)$ for any $\alpha > 0$, then we can consider e_i defined all over R^n_{++} by $e_i(\alpha\lambda) = e_i(\lambda)$ for all $\lambda \in \Delta^{n-1}_{++}$.*

3 Equilibrium and Excess Utility Function.

Let us now consider the following problem:

$$\begin{array}{c} \max_{x\in\mathcal{M}} \sum_i \bar{\lambda}_i \int_\Omega U_i(s,x_i(s))d\nu(s) \\ \text{subject to } \sum_i x_i(s) \le \sum_i w_i(s) \text{ and } x_i(s) \ge 0. \end{array} \tag{6}$$

Is a well known proposition that for an allocation $\bar{x}$, is Pareto optimal if and only if we can choose a $\bar{\lambda}$, such that $\bar{x}$ solves the above problem, with $\lambda = \bar{\lambda}$. Moreover, since a consumer with zero social weight receive nothing of value at a solution of this problem, we have that if $\bar{x}$ is a strictly positive allocation, that is $\{\bar{x} \in R^l_{++}\}$, all consumption has a positive social weight. See for instance [Ke]. Reciprocally if $\bar{\lambda}$ is in the interior of the simplex, then from remark (1) the solution $x(.,\lambda)$ of (6) is a strictly positive Pareto optimal allocation. (This is guaranteed also by the following boundary condition on preference: $\{v(s) \in R^l_{++} : v(s) \succeq_i w_i(s)\}$ is closed for a.e.s, for all i and $w_i(s)$ strictly positive.)

From the first theorem of welfare, we have that every equilibrium allocation is Pareto optimal.

Let $(\bar{x})$ be an equilibrium allocation, then there exists a $\bar{\lambda}$ such that $\bar{x} = \{\bar{x_1}, \ldots, \bar{x_n}\} : \Omega \to R^n$, is a solution for the problem in the beginning of this section.

In the conditions of our model, the first order conditions for this problem are the same that for (3). Then if a pair $(\bar{p}, \bar{x})$ is an price-allocation equilibrium, there exists a $\bar{\lambda}$ such that $\bar{x}(s) = \bar{x}(s, \bar{\lambda})$; solves (6) and $\bar{p}(s) = \gamma(s, \bar{\lambda})$, solves (4) for a.e.s.

Moreover we have the following proposition:

Proposition 2 *A pair $(\bar{p}, \bar{x})$ is an equilibrium, if and if there exists $\bar{\lambda} \in \triangle^{n-1}$ such that $\bar{x}(s) = \bar{x}(s, \bar{\lambda})$; solves (6), and $\bar{p}(s) = \gamma(s, \bar{\lambda})$, solves (4) for a.e.s and $e(\bar{\lambda}) = 0$.*

Proof: Suppose that $\bar{x}(\cdot, \bar{\lambda})$ solves (6) and $\gamma(s, \bar{\lambda})$ solves (4). If for $\bar{\lambda} \in \triangle^{n-1}$, we have that $e(\bar{\lambda}) = 0$, then the pair $(\bar{p}, \bar{x})$, with $\bar{p} = \gamma(\cdot, \bar{\lambda})$ and $\bar{x} = x(\cdot, \bar{\lambda})$, is an equilibrium.

Reciprocally, if $(\bar{p}, \bar{x})$ is an equilibrium, then is straightforward from definition that $e(\lambda) = 0$. From de first welfare theorem, there exists $\bar{\lambda} \in \triangle^{n-1}$, such that $\bar{x}$ is a solution for (6). Since p is a equilibrium price, it is a support for $\bar{x}$, i.e. if for some x we have that $u_i(x) \ge u_i(\bar{x}), i = \{1, ..., n\}$, strictly for some i then $\langle \bar{p}, x_i \rangle > \langle \bar{p}, w_i \rangle$ and from the first order conditions we have that: $\bar{p}(s) = \gamma(s)$. The proposition is proved.

Let be $\triangle^{n-1}_{++} = \{\lambda \in \triangle^{n-1} : \lambda_i > 0 \; \forall \; i = 1, \cdots, n\}$.

We will now the definition of the equilibrium set.

Definition 8 *We will say that λ is an equilibrium for the economy if $\lambda \in E$, where $E = \{\lambda \in \triangle^{n-1}_{++} : e(\lambda) = 0\}$. The set E will be called, the equilibrium set of the economy.*

4 A Binary Relation In The Social Weights Space

Let $e : R^n \to R^n$ be a excess utility function.

Let us define $\succ$ in $\bar{\triangle}_\epsilon^{n-1} = \{\lambda \in R_+^n : \sum_{i=1}^n \lambda^i = 1; \lambda_i \geq \epsilon\}$ a subset of the social weights space.

Definition 9 *We define $\succ$ as:*

$$(\lambda_1, \lambda_2) \in \succ \quad iff \quad \lambda_1 e(\lambda_2) < 0.$$

We will write $\lambda_1 \succ \lambda_2$.

Properties of the Binary Relation $\succ$.

$\succ$ is irreflexive, convex, and upper semi-continuous.

- *irreflexive* $\lambda \not\succ \lambda$ because $\lambda.e(\lambda) = 0$.
- *convex* if $\lambda^1 \succ \lambda$ and $\lambda^2 \succ \lambda$, then $\alpha\, \lambda^1 + \beta\, \lambda^2 \succ \lambda$ with $\alpha + \beta = 1$.
- *upper semi-continuous,* $A = \{\alpha \in \triangle_\epsilon^{n-1};\ \lambda \succ \alpha\}$ is open

  Proof:

  $$A = \{\alpha \in \triangle_\epsilon^{n-1};\ \lambda.e(\alpha) < 0\},$$

  by the continuity of $\lambda.e(.)$, exist an open neighborhood V_α of α, such that $\lambda.e(V_\alpha) < 0$. Then A is open.

5 Existence of Equilibrium.

Definition 10 *We say that γ is a maximal element of $\succ$ if there does not exist λ such that a $\lambda \succ \gamma$.*

Lemma 2 *The set of maximal elements in $\bar{\triangle}_\epsilon^{n-1}$ is non-empty.*

Proof: Note that

$$F(\lambda) = \bar{\triangle}_\epsilon^{n-1} - \{\alpha \in \bar{\triangle}_\epsilon^{n-1} \text{ such that } \lambda \succ \alpha\} = \{\alpha \in \bar{\triangle}_\epsilon^{n-1} \text{ such that } \lambda.e(\alpha) \geq 0\}$$

is a compact set.

We can also see that the convex hull of $\{\lambda_1, \cdots, \lambda_k\}$ is contained in $\cup_{i=1}^k F(\lambda_i)$ for all finite subset $\lambda_1, \cdots, \lambda_k \in \bar{\triangle}_\epsilon^{n-1}$. To this end let be $\lambda_1, \ldots, \lambda_k \in \bar{\triangle}_\epsilon^{n-1}$. If $\gamma = \sum_{i=1}^k a_i \lambda_i$ is a convex combination and γ is not in $\cup_{i=1}^{k} F(\lambda_i)$, then $\lambda_i \succ \gamma$ for every $i = 1, \ldots, n$, and so, since $\succ$ is convex value, we must have $\gamma \succ \gamma$. This is not possible because $\succ$ is irreflexive.

Then from Fann-Theorem, (see for instance [BC]) it follows that $\cap_{\lambda \in \bar{\triangle}_{\epsilon}^{n-1}} F(\lambda) \neq \emptyset$. It is easy to see that the set of maximal elements in $\bar{\triangle}_{\epsilon}^{n-1}$ is equal to $\cap_{\lambda \in \bar{\triangle}_{\epsilon}^{n-1}} F(\lambda)$.

Then the theorem follows.

Theorem 1 *Let $\mathcal{E}$ be an economy with infinite dimensional consumption space, with differentiable strictly convex C^2 and separable utilities. Then $\mathcal{E}$ has a non-empty, compact set of equilibrium.*

Proof: From lemma 2 we know that there exists γ_{ϵ_n} a maximal element in $\bar{\triangle}_{\epsilon_n}^{n-1}$. The collection $\{\bar{\triangle}_{\epsilon_n}^{n-1}\}$ may be directed by inclusion. Consider $\epsilon_n \to 0$, and $\gamma_{\epsilon_n} \in \bar{\triangle}_{\epsilon_n}^{n-1} \subset \bar{\triangle}^{n-1}$, since $\bar{\triangle}^{n-1}$ is a compact set, there exists $\gamma \in \bar{\triangle}^{n-1} = \{\lambda \in R_{+}^{n} \sum_{i=1}^{n} \lambda_i = 1\}$ and a subnet $\{\gamma_{\epsilon'_n}\}$ such that $\gamma_{\epsilon'_n} \to \gamma$. If we prove that: $\gamma \in \triangle_{++}^{n-1} = \{\lambda \in \bar{\triangle}^{n-1}, \text{and } \lambda >> 0\}$ and that $e(\gamma) = 0$, then the theorem follows. Suppose that $\gamma \in \partial\bar{\triangle}^{n-1} = \{\lambda \in \bar{\triangle}^{n-1} \text{and at least one } \lambda_i = 0 \ i \in \{1, \cdots, n\}\}$. Is straightforward from the definition that $\lim_{\lambda \to \partial\bar{\triangle}^{n-1}} \|e(\lambda)\| = \infty$ since e is bounded above, see lemma 2), then there exists $\xi \in \triangle_{++}^{n-1}$ and ϵ_0 such that $\xi e(\gamma_{\epsilon''}) < 0, \forall \epsilon^{''} < \epsilon_0$. Since $\xi \in \bar{\triangle}_{\epsilon'''}^{n-1}, \forall \epsilon^{'''} < \epsilon_0^{'} \leq \epsilon_0$, the last inequality contradicts the maximality of $\gamma_{\epsilon''}$.

Suppose now there exists a $e_i(\gamma) < 0 \ i = \{1, \cdots, n,\}$ then for same $\xi \in \triangle_{++}^{n-1}$ we have that $\xi e(\gamma) < 0$. From the continuity of $\xi e(\cdot)$ we obtain that $\xi e(\gamma_{\epsilon'_0}) < 0, \forall \epsilon'_0 > \epsilon_0$, this contradicts the maximality of $\gamma_{\epsilon'_0}$. Then $e(\gamma) \geq 0$ follows. Since $\gamma \in S$ and $\gamma e(\gamma) = 0$, then $e(\gamma) = 0$.

The theorem is proved.

Then the set $E = \{\lambda : e(\lambda) = 0\}$ is non empty. That is, there exists at least one equilibrium $(x(s, \lambda), p(s, \lambda))$ for $\mathcal{E}$. .

6 Uniqueness From W.A.R.P.

Let us now to define the weak axiom of revealed preference (W.A.R.P.) from the excess utility function.

Definition 11 *We say that the excess utility function satisfies the weak axiom of revealed preference (WARP) if*

$$\lambda_1.e(\lambda_2) \ \geq \ 0 \ \textit{ then } \ \lambda_2.e(\lambda_1) \ < \ 0$$

Theorem 2 *WARP implies uniqueness of equilibrium.*

Proof: We argue by contradiction. Suppose that λ_1 and λ_2 are two equilibria.

From Proposition 6) we have that $e(\lambda_1) = e(\lambda_2) = 0$.

Then $\lambda_i e(\lambda_j) = 0$, thus W.A.R.P. yield the following inequality $\lambda_j e(\lambda_i) < 0, i = \{1, 2\}, j = \{1, 2\}$.

Uniqueness follows.

Definition 12 *Let e be a excess utility function, then e is monotone on* $T_\lambda = \{\bar{\lambda} \in R^n : \bar{\lambda}\lambda = 0\}$ *if* $(\lambda_1 - \lambda_2)(e(\lambda_1) - e(\lambda_2)) > 0$, *whenever* $(\lambda_1 - \lambda_2) \in T_\lambda, e(\lambda_1) \neq (\lambda_2)$.

Proposition 3 *If* $(e(\cdot))$ *is a monotone function,* $e(\cdot)$ *has W.A.R.P.*

Proof: Suppose that $\lambda_2 e(\lambda_1) \geq 0$. Since $\lambda_i \lambda > 0; i = 1, 2$, there exists $\alpha > 0$ such that $\lambda_1 - \alpha\lambda_2 \in T_\lambda$. Hence $(\lambda_1 - \alpha\lambda_2)(e(\lambda_1) - e(\alpha\lambda_2)) > 0$, follows, and then $-\lambda_1 e(\alpha\lambda_2) > \alpha\lambda_2 e(\lambda_1) \geq 0$. Since e is a homogeneous degree zero function, $\lambda_1 e(\lambda_2) < 0$. We have concluded our proof.

6.1 Some Applications

Proposition 4 *If the central planner chooses* λ *using the rule* $\succ$, *and if the excess utility function has WARP, then the* λ *selected by the central planner is an equilibrium.*

From WARP we have that $\bar{\lambda} e(\lambda) < 0$. That is $\bar{\lambda} \succ \lambda$.

Economies with WARP in the Excess Utility Function

Example 1 *Suppose an economy with the following utility functions:*

$U_i(x) = x(s)^{\frac{1}{2}}$, *endowments* $w_1(s) = as$ *and* $w_2(s) = (1-a)s$, *with* $0 < a < 1, s \in (0, 1)$ *and* μ *the Lebesgue mesure.*

The excess utility function is,

$$e(\lambda) = \left\{ \int \frac{1}{2} x_1^{-\frac{1}{2}} (x_1 - w_1) d\mu(s), \int \frac{1}{2} x_2^{-\frac{1}{2}} (x_2 - w_2) d\mu(s) \right\}$$

From the first order condition:

$$x_i(s) = \frac{\lambda_i^2}{\lambda_1^2 + \lambda_2^2} s$$

Substituing in the above equation we obtain that:

$$e(\bar{\lambda}) = 0, \text{ iff } \bar{\lambda} = \left\{ \frac{\sqrt{a}}{\sqrt{a} + \sqrt{1-a}}, \frac{\sqrt{1-a}}{\sqrt{a} + \sqrt{1-a}} \right\}.$$

Is ease to see that :

$$\bar{\lambda} e(\lambda) < 0 \ \forall \ \lambda, \text{ i.e. } \bar{\lambda} \succ \lambda.$$

Example 2 *For economies with utilities* $U_i(x) = Lgx, i = (1, 2)$ *we obtain WARP in the excess utility function.*

7 Concluding Remarks.

In economies with infinite dimensional consumption spaces, the agent's budget may not be compact. Hence the existence of demand function need not be a consequence of the utility maximization problem. In our approach without assuming its existence, with a simple proof, we have obtained the existence of the competitive equilibrium. So the excess utility function appears as a powerful tool in order to obtain a deeper insight in the structure of the equilibrium set. Some additional assumptions about the behavior of the excess utility function allow us to obtain a sufficient condition for uniqueness of the Walrasian equilibrium. Unfortunately its economical interpretations are not straightforward.

Bibliography

[BC] Baiochi, C., Capelo, A. (1989) "Variational and Quasi Variational Inequalities." *John Wiley and Sons ltd.*

[Ke] Kehoe, T.,J.(1991) Computation and Multiplicity of Equilibria", *Handbook of Mathematical Economics*, **vol IV**. Edited by Hildenbrand, W. and Sonnenschein, H.

[MC] Mas-Collel (1989) "Indeterminacy in Incomplete Market Economies" *Economic Theory* **1** 45-62.