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A characterization of the singular economies of the infinite dimensional models in general equilibrium theory

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A Characterization of the Singular Economies of the Infinite Dimensional Models in General Equilibrium Theory.*

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Abstract

The aim of this paper is to characterize the set of singular economies, when there are a finite set of consumers with infinitely many goods in the sense that goods differ in the time which they are consumed or in the state of the world in which they become available. There exist l available goods in each time or in each state of the world.

Employing well know results of the " Singularity Theory" on differentiable maps, we characterize the structure of the equilibrium set from the so called singular economies.

In the last section we introduce a continuous time economy and -although in a limited way- we study the dynamics along the equilibrium path. We show that if there exist singularities then the equilibrium set isn't a finite set, moreover it may be a continuous set of equilibria.

It is important to notice that we will not describe our models in terms of the tâtonnement. The process of endowments move the price system, we don't need the demand law to characterize the equilibrium manifold.

1   Introduction

A large part of the results of General Equilibrium Theory can be summarized by saying that the equilibrium set does not show qualitative changes, as long as the initial endowments vary within the same connected component of the set of regular economies. Changes in the number of equilibria can be observed only when initial endowments are varied in such a way that they come across singular economies, in the sense that we will define later. Our improvement to understand

the nature of these changes in infinite dimensional models requires an investigation in the structure of the equilibrium manifold, with special attention to their singularities.

It is well known that the demand function is a good tool to deal with the equilibrium manifold in economies in which consumption spaces are subset of R^l [Mas-Colell, A. (1985)], but unfortunately if the consumption spaces are subsets of infinite dimensional spaces, the demand function may not exist, see [Araujo, A. (1987)]. However it is possible to characterize the equilibrium set from the excess utility function, see for instance [Accinelli, E. (1996)].

In [Balasko, Y. 1997a] and [Balasko, Y. 1997b] the submanifold of regular equilibria for infinite dimensional economies is characterized using the natural projection method. In this paper we will study the submanifold of singularities using the excess utility function for models with infinite many goods. We will show that equilibrium manifold for these models has analogous properties to the equilibrium manifold for models with a finite quantity of goods. On the other hand, we obtain some properties of the equilibrium submanifold of singular economies. We employ the modern classification of singularities to characterize this equilibrium submanifold in some special cases.

Finally we will describe an inter-temporal model of perfect foresight with a continuous set of equilibrium paths.

2 The Model

In our work commodities are defined as physical goods which may differ in time at which will be consumed or in the state of the world in which they become available. As we allow an infinite variation in any of these contingentes, we consider economies with infinitely many goods and a finite number of agents. We take as primitive $\{S, \mathcal{F}, \mu\}$ a measure space, where S is the set of the possible states how the world will be tomorrow, or is a time interval.

The characteristics of the economic model in this paper are the following:

i) The commodity-price system will be described by a dual pair (x, γ) of the dual system (B, B^*). There are l goods available for consumption in each state of the world or in each time. Commodity space B is a product space of the l $L_q(\mu)$ functional spaces, and B^* is the product of the l, $L_p(\mu)$ dual spaces of the prices, where $\frac{1}{q} + \frac{1}{p} = 1$.

As usual the evaluation $< x, \gamma >$ is the natural duality function, for all $x \in B$ and $\gamma \in B^*$.

ii) There are n consumers indexed by i such that:

ii.1) Each consumer i has the positive cone B^+ as his consumption set.

ii.2) Each consumer has an initial endowment

$$w_i = (w_{i1}, w_{i2}, ..., w_{il}), \quad w_{ij} \in L_q^+(\mu), w_{ij} \geq 0, w_{ij} \neq 0, \quad \forall j = 1, 2, ...l.$$

Endowments, $w_i \in \mathcal{M}$ are bounded from above and bounded away from zero in every component, i.e. there exists, h and H with h $< w_{i_j}(s) <$ H for each $j = 1 ... l$, and $s \in \Omega$.

The total endowment of the economy will be denoted by w, i.e. $w = \sum_{i=1}^{i=n} w_i$.

ii.3) The preference of each consumer i are represented by a monotone quasi concave utility function: $U_i : B^+ \to \Re$, given by $U_i(x) = \int_S u_i(s, x(s)) d\mu(s)$. Monotonicity means, of course, that $x > y$ (i.e., $x_j(s) \geq y_j(s), x \neq y, \ j = 1, 2, ..., l$; in almost every way (a.e.) s) implies $U_i(x) > U_i(y)$.

We suppose that the following regular conditions for the utility functions are satisfies:

a) Each $u_i(s, \cdot)$ is a strictly concave, C^2 function, for a.e. s, and satisfy the Inada condition, i.e.: $|grad \ u_i(s, 0)| =: lim_{x_j \to 0} |grad \ u_i(s, x)| = \infty, \ j = \{1, 2, ..., l\}$.

b) For each $u_i : S \times B^+ \to \Re$ there exist $a_i \in (L_p^+(\mu))^l$ and $b_i \in (L_1^+(\mu))^l$ such that: $u_i(s, x(s)) \leq a_i(s)x(s) + b_i(s)$ for every $(s, x(s)) \in S \times \Re^l$.

c) We will consider in the space V of all measurable functions $u : S \times R_{++}^l \to R$, the topology of the C^2 convergence on compacta, where

$$\|u\|_K = ess \sup_{s \in \Omega} \max_{z \in K} \left\{ u(s, z)| + |\partial u(s, z)| + |\partial^2 u(s, z)| \right\}.$$

We will assume that all the $u(s, \cdot)$ belong to a fixed compact subset Λ of V.

Recall that a real number M, is the essential supremun of f, and we write, $ess \sup_{s \in \Omega} f(s)$ if $|f| \leq M$ for almost all $s \in \Omega$.

Remark 1 *In this paper utilities are fixed and each economy will be characterized by the endowments. The utility functions $u_i(s, .)$ belongs to a fixed compact subset Λ, of V for each $s \in \Omega$ and $u_i \in V$.*

It is well know that in the classic Arrow-Debreu model the equilibrium manifold is:

$$\mathcal{E} = \{(p, w) \in \Pi^{l-1} \times \Omega : z(p, w) = 0\},$$

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where $z : \Pi^{l-1} \times \Omega \to \Re^l$ is the excess demand function, $\Pi^{l-1} = \{p \in \Re^l; \sum_{i=1}^l p_i^2 = 1\}$ and $\Omega = \Re^{ln}$.

It is important to insist that the demand function couldn't exist in economies with infinite dimensional commodity space, hence the characterization of the equilibrium manifold in the sense of the standard Arrow-Debreu model cannot be extended to the infinite dimensional case.

To obtain the equilibrium manifold we follow the Negishi approach.

The sum of the weighted utilities of the agents is maximized subject to the resource constraint:

$$\max_x \sum_{i=1}^n \lambda_i U_i(x); \ \ s.t. \ \sum_{i=1}^n x_i(s) \le w(s); \ \forall \ s \ \in \ S.$$

The solution to this constrained optimization problem (or Negishi problem) determines implicit vector prices $\gamma(s, \lambda)$, i.e. the Lagrange multipliers at the solution x in the consumption set.

Let us define the excess utility function, $e : \triangle^{n-1} \times \Omega_\infty \to \Re^n$:

$$e(\lambda, w) = \left\{ \frac{1}{\lambda_1} \int_S \gamma(s, \lambda)(x_1(s, \lambda) - w_1(s))d\mu(s), \dots, \frac{1}{\lambda_n} \int_S \gamma(s, \lambda)(x_n(s, \lambda) - w_n(s))d\mu(s) \right\},$$

where $\triangle^{n-1} = \{\lambda : \lambda \in \Re^n, \sum_{i=1}^n \lambda_i = 1, \lambda_i > 0, i = 1, 2, \dots n\}$ is the $n-1$ dimensional simplex, $\Omega_\infty = \times_{i=1}^l B^+$, and $\gamma(\cdot, \lambda)$ is the Lagrange multiplier associated with the optimization Negishi problem.

Then the Pareto optimality of a walrasian equilibrium is invoked to establish that the set of walrasian equilibria is in one-to-one correspondence with the zeros of the excess utility function, see [Accinelli, E. (1996)]

In the conditions of this model the next proposition follows.

Proposition 1 *A pair (γ, x) is an equlibrium if and only if there exists $\lambda \ \in \triangle^{n-1}$ such that: $x(s, \lambda)$ solve the Negishi problem and $\gamma(s, \lambda)$ is the Lagrange multiplier for this problem.*

The proposition is proved in [Accinelli, E. (1996)].

The proposition proves that for infinite dimensional economies, with endowments in B, and preferences as above the equilibrium set may be represented by :

$$\mathcal{E}_\infty = \{(\lambda, w) \in \Delta^{n-1} \times \Omega_\infty : e(\lambda, w) = 0\},$$

3   The Equilibrium Manifold

In this section we prove that $\mathcal{E}_\infty$ is a Banach manifold. In the process of the demonstration, it will show that the excess utility function play the same role that the excess demand function for a standard Arrow-Debreu model with a finite number of goods.

For general mathematical references and definitions we cite [Zeidler, E. (1993)].

Lemma 1 *Assume that utility functions satifies conditions a), b), c), then the excess utility function e is differentiable with respect to λ in the interior of $\triangle^{n-1}$.*

Proof:

It suffices to prove the differentiability of e with respect to λ on subset of $\triangle^{n-1}$ away from zero, that is $\lambda_i > 0$, for every $i = 1, 2, ...n$. Applying the implicit function theorem to the first order conditions, it follows that $x_{ij}(s, \cdot)$ is differentiable for every s, see [Accinelli, E. (1996)].

Let $E_{ij} = \frac{1}{\lambda_i} \gamma_j (x_i - w_i)$, from the first condition in the Negishi problem, it follows

$$E_{ij}(s, \lambda) = \frac{\partial u_i(s, x_i)}{\partial x_j}(x_i(s, \lambda) - w_i(s)), \ i = 1, 2, ...n, \ j = 1, 2, ..., l$$

Taking derivatives we obtain:

$$\frac{\partial E_i(s, \lambda)}{\partial \lambda_k} = \frac{\partial x_i}{\partial \lambda_k} \left\{ \partial^2 u_i(s, x_i(s, \lambda))[x_i(s, \lambda) - w_i(s)]^{tr} + [\partial u_i(s, x_i(s, \lambda))]^{tr} \right\},$$

where tr is the symbol of transposition, and

$$\partial^2 u_i = \begin{bmatrix} \partial^2 u_i / \partial x_1^2 & \partial^2 u_i / \partial x_1 \partial x_2 & \cdots & \partial^2 u_i / \partial x_1 \partial x_l \\ \partial^2 u_i / \partial x_2 \partial x_1 & \partial^2 u_i / \partial x_2^2 & \cdots & \partial^2 u_i / \partial x_2 \partial x_l \\ \vdots & \vdots & & \vdots \\ \partial^2 u_i / \partial x_l \partial x_1 & \partial^2 u_i / \partial x_l \partial x_2 & \cdots & \partial^2 u_i / \partial x_l^2 \end{bmatrix}.$$

Then from item c, the Remark 1 (below c), and the Lebesgue dominated convergence theorem it follows that e is differentiable with respect to λ and its derivative is:

$$\frac{\partial e_i(s, \lambda)}{\partial \lambda_k} = \int_S \frac{\partial x_i}{\partial \lambda_k} \left\{ \partial^2 u_i(s, x_i(s, \lambda))[x_i(s, \lambda) - w_i(s)]^{tr} + [\partial u_i(s, x_i(s, \lambda))]^{tr} \right\}.[]$$

Remark 2 *Recall that 0 is a regular value of $e : \triangle^{n-1} \times \Omega_\infty \rightarrow \Re^n$ if e is a submersion for all (λ, w) such that $e(\lambda, w) = 0$, i.e. the dimension of the rank of the jacobian matrix $[J_\lambda e(\lambda, w)]$ for $(\lambda, w) \in \mathcal{E}_\infty$ is $n - 1 : dim(rank[J_\lambda e(\lambda, w)]) = n - 1$.*

Definition 1 *We call R the set of regular points (λ, w), if 0 is a regular value for the excess utility function $e(\lambda, w)$ with $\lambda \in \triangle^{n-1}$, the $n - 1$ dimensional simplex and $w \in \Omega_\infty$. The complementary set of R, the critical points set, will be noted by $C = \mathcal{E}_\infty - R$.*

Theorem 1 *Let R be the regular points set. Then:*

1) R is an open and dense set in $\triangle^{n-1} \times \Omega_\infty$.

2) The equilibrium set, $\mathcal{E}_\infty = \{(\lambda, w) \in \triangle^{n-1} \times \Omega_\infty : e(\lambda, w) = 0\}$, is a Banach manifold.

Proof:

- 1)See [Mas-Colell, A. (1990)].

- 2) Consider $e(\cdot, w) : \triangle^{n-1} \to \Re^n$

    i) For each parameter, $w \in \Omega_\infty$, $e(\cdot, w)$ is a Fredholm Operator of index cero, because: $J_\lambda e(\cdot, w) : \Re^{n-1} \to \Re^{n-1}$ and $dim(ker\,[J_\lambda e(\cdot, w)]) = codim(rank\,[J_\lambda e(\cdot, w)]) = 0$.

    ii) Convergence $e(\lambda_m, w_m) \to 0$ as $m \to \infty$ and convergence of (w_m) in Ω_∞ implies the existence of a convergent subsequence of λ_m in $\triangle^{n-1}$. It is sufficient consider only compact subsets in $\triangle^{n-1}$ bounded away from zero. Then existence of a convergent subsequent $\{\lambda_m\}$ follows from the compactness of the subset considered.

    Recall that in equilibrium, there is not any λ in the simplex with cero in any of its coordinates.

Then following [Zeidler, E. (1993)],vol.1 pag. 189, we obtain that the solution set of

$$e(\lambda, w) = 0 \quad \lambda \in \triangle^{n-1}, w \in \Omega_\infty,$$

is a Banach manifold.[]

Corollary 1 *The set of critical points C is closed and has empty interior.*

Proof: As the regular points is open and dense, his complement C satisfies this corollary[].

In the next section from the excess utility function, we will analyze the submanifold of critical points in the equilibrium set.

4 Singularities on the Equilibrium Manifold

Regular values of maps are related with situations where infinitesimal variations of the arguments, entails infinitesimal variation of the values. The number of preimages are locally constant for regular values, changes in the number of preimages are observed only in a neighborhood of a singular value.

The changes in the number of equilibrium are observed only when endowments come across w in the critical points set. As application of the Sard-Smale theorem we will prove that this set is meager and rare in Ω_∞.

Nevertheless this singular set play a very special role, when the endowments w cross these points each equilibrium leads to several new equilibrium forms (the bifurcation case). In this points we obtain abrupt changes in prices or in social weights.

This apparently inexplicable and unpredictable discontinuity leads to the serious, sometimes heated question of the market mechanism, and even to irrational behaviour that occasionally ends in widespread destruction of resources through futile attempts to get back to the former price levels.

In this section we show that the excess utility function allow us to transform the infinite dimensional problem stated above in a finite dimensional one, this approach allow us obtain in a natural way, a deeper insight in the structure of the equilibrium set in the infinite dimensional case. We work like [Balasko, Y. 1997b] say, "to draw an almost perfect parallel between the finite an infinite dimensional models in terms of the properties of the natural projection", but in our case is done in terms of the equilibrium manifold, permitting us to work with differential techniques in General Equilibrium Theory.

We prove that:

i) For each singular economy w the set of critical points (or $\lambda-$singularities) in the equilibrium manifold is a submanifold with dimension less than $n - 1$.

ii) The set above mentioned has zero measure.

As it is well know for finite dimensional cases there exist the same kind of results. As we will show they are valid for both finite and infinite dimensional cases.

We recall the following mathematical facts:

a) Let X and Y be smooth manifold of the dimension n and $f : X \to Y$ be a differentiable mapping, the set of singularities of corank k of f is

$$S_k(f) = \{\bar{x} \in X : dim([rank J_x f(\bar{x})]) = n - k\}.$$

b) The set k - jet $\mathcal{J}^k(X, Y)$ is the family of the equivalence classes: $\mathcal{C}_f, \mathcal{C}_g, \ldots$, of differentiable functions, $f, g, \ldots$ such that for all f_1, f_2 belong to $\mathcal{C}_f$ hold that $f_1(x) = f_2(x)$ and $J f_1(x) = J f_2(x)$ has $k - 1$ order contact.

c) The one jet of f is: $jf : X \to \mathcal{J}(X, Y)$.

d) Given $S_k(f)$, its image under the one-jet of f

$$jf[S_k(f)] = \mathcal{S}_k \subset \mathcal{J}(X, Y)$$

is the equivalence class of function f of which singularities are corank k. Recall that $\mathcal{S}_k$ is a submanifold of $\mathcal{J}(X, Y)$.

e) f is one-generic if the one jet of f is transversal to $\mathcal{S}_k$ for all k.

From the Thom Transversality Theorem the set of *one generic* is residual in the Whitney Topology. Recall that a set is residual if contains a countable intersection of open dense sets, see [Golubistki, M. and Guillemin,V.(1973)].

The above facts able us to set some properties of the excess utility function.

Definition 2 *1) Let $e_w(\cdot) = e(\cdot, w) : \triangle^{n-1} \to \Re^{n-1}$, with images $e_w(\lambda) = y$ and the jacobian of this function evaluated in λ is: $Je_w(\lambda) : T_\lambda \triangle^{n-1} \to T_{e_w(\lambda)} \Re^{n-1}$ where T denote the tangent space.*

2) Let $SE_k(e_w(\cdot) = 0)$ be the set of social weights λ where $dim\,(rank\,[J_\lambda e(\lambda, w)]) = n-1-k$ and $SE(w) = \cup_{k=1}^{n-1} SE_k(e_w(\cdot) = 0)$ then $SE = \cup_{w \in \Omega_\infty} \{w : SE(w) \neq \Phi\}$ is the set of singular economies that belong to the set of exchange economies E.

3) The one-jet of the excess utility function is $je_w : \triangle^{n-1} \to \mathcal{J}\left(\triangle^{n-1}, \Re^{n-1}\right).$

4) The set image of $SE_k(e_w(\cdot) = 0)$ by the one-jet of the excess utility function will be denoted by $\mathcal{SE}_k \subset \mathcal{J}\left(\triangle^{n-1}, \Re^{n-1}\right).$

From the above definitions it follows that for all $k, SE_k(e_w(\cdot) = 0) = \triangle^{n-1} \cap C$ and $SE = \Omega_\infty \cap C$. It is important to recall at this point, like any exchange economy, each singular economy w is a vectorial field depending of a state variable, like time or states of the world.

We prove in the next theorem that $SE_k(e_w(\cdot) = 0)$ is a submanifold. The notation and the aim of the proof follows from [Golubistki, M. and Guillemin,V.(1973)].

Theorem 2 *Let e_w be one generic map, then the set of $\lambda-$singularities, $SE_k(e_w(\cdot) = 0)$ with corank $[Je_w] = k$ is a submanifold with*

$$dim\,(SE_k(e_w(\cdot) = 0)) < (n-1) - k^2.$$

Proof: Let $J(\triangle^{n-1}, \Re^{n-1})$ the equivalence class of e_w for mappings $f : \triangle^{n-1} \to \Re^{n-1}$ under the one order of contact. From the fact that e_w is one generic, the set of $\lambda-$ singularities,

$$SE_k(e_w(\cdot) = 0) = (je_w)^{-1} [\mathcal{SE}_k]$$

is a submanifold with $codim\,(SE_k\,(e_w(\cdot) = 0)) = k^2$. []

Remark 3 *From the definition above it follows that $SE(w)$ and SE are stratified sets.*

Theorem 3 *The set of singular economies SE is a subset of zero measure.*

Proof: It follows from the fact that every stratum $SE_k(e_w(\cdot) = 0)$ is a submanifold with strictly positive codimension, therefore it has measure zero. As $S(w)$ is a finite union of sets of measure zero, itself has measure zero. Moreover the infinite union of $S(w)$ is also a set of zero measure.[]

Corollary 2 *If $w \in SE$ then the set $SE(w)$ has an empty interior, and each one of his strata $SE_k(e_w(\cdot) = 0)$ too.*

Proof It follows straightforward from the fact that all these sets have zero measure .[]

Following [Golubistki, M. and Guillemin,V.(1973)] we say that the subspaces $H_1, H_2, ...H_k$ of $\Re^{n-1}$ are in *General Position* if and only if, given $v_1, v_2, ...v_k$ in $\Re^{n-1}$, there exists $h_i \in H_i$ and $z \in \Re^{n-1}$ such that $v_i = z + h_i$ for all i.

Definition 3 *We say that the subspace H is defined by $\lambda \in e_w^{-1}(0)$ if $H = [Je_w|(\lambda)]\,T_\lambda \triangle^{n-1}$*

Theorem 4 *For each singular economy given by w the set of $\lambda-$singularities in $e_w^{-1}(0)$, that define subspaces H in general position, has at most $n - 1$ points.*

Proof: We shall argue by contradiction. Suppose that $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ consists of distinct $\lambda-$singularities of $e_w^{-1}(0)$ that define subspaces in general position. Let $H_i = [Je_w|(\lambda_i)]\,T_{\lambda_i} \triangle^{n-1}$. Thus, from [Golubistki, M. and Guillemin,V.(1973)], lemma 1.7, $n-1 \geq codim(H_1 \cap ... \cap H_n) = \sum_{i=1}^{n} codim H_i \geq n$. The last inequality holds since if λ_i is a $\lambda-$singularity and because the codim$H_i = codim\,([Je_w(\lambda_i)]\,T_{\lambda_i} \triangle^{n-1}) \geq 1$.[]

5   Some Kinds of Singularities

In this section we show that different economic models have the same kind of singularities. Regular economies have a similar behaviour. Singular economies are characterize by the fact that small variation in the initial endowments cause sudden changes in equilibrium social weights. Then for economic models with the same kind of singularities we will be able to observe, from time to time, the same critical behavior.

From these observations and the modern theory of singularities, we are able to obtain a classification of the economies according to the kind of the singularities.

The next theorem characterize the singularities in a general but simple economic model.

Theorem 5 *For a given singular economy w with two agents, the set of $\lambda-$singularities is a 0-dimensional submanifold, i.e. these singularities are isolates points.*

Proof: From the theorem 2 it follows that $dim\left(SE_k(e_w(\cdot) = 0)\right) < 1.$ []

We recall the following statement:

Let $\mathcal{L}_r(X, Y)$ be the space of linear maps of X into Y which drop rank by r. Then $\mathcal{L}_r(X, Y)$ is a submanifold of the homomorphisms $\mathcal{H}(X, Y)$ of codimension $r^2 + ar$ where $a = |dimX - dimY|$.

Theorem 6 *Given an economy with n agents and*

$$k > \sqrt{n-1}. \tag{1}$$

then the set $SE_k(e_w(\cdot) = 0)$ is empty.

Proof: The codimension of $SE_k(e_w(\cdot) = 0)$ is equal to the codimension of $\mathcal{L}_k(\triangle^{n-1}, \Re^{(n-1)})$ in $\mathcal{H}(\triangle^{n-1}, \Re^{(n-1)})$, it follows that $\dim[SE_k(e_w(\cdot) = 0)] = k^2 + ak$, where $a = dim(\triangle^{n-1}) - (n-1) = 0$. Then if eq. 1 we hold that $(n-1) - k^2 < 0$, then the $\text{codim}[SE_k(e_w(\cdot) = 0)] < 0$ and for this contradiction it follows that the set is empty. []

Example 1 *In a model of pure exchange with 2 agents, there is no singular economies with $k > 1$.*

Substituing in 1 the result follows.[]

Example 2 *Now we will consider an economy with three agents, suppose that the endowments $w = \{w_1, w_2, w_3\}$ are fixed. In this case the excess utility function $e = \{e_1, e_2, e_3\} : \triangle^2 \rightarrow \Re^2$, where $\triangle^2 = \{\lambda \in \Re^3_+ : \lambda_1 + \lambda_2 + \lambda_3 = 1\}$. Then e is a map between 2-manifolds. Their critical points set is a submanifold, and $e|SE_k(e_w(\cdot) = 0)$ is again a map between manifolds. By our computation S_1 has codimension 1 in $\triangle^2$, and $SE_2(e_w(\cdot) = 0)$ does not occur since it would have to be of codimension 4.*

Then from [Golubistki, M. and Guillemin,V.(1973)] we know that generically only one of the following two situations can occur:

Let $\bar{\lambda}$ be in $SE_1(e_w(\cdot) = 0)$.

a) $T_{\bar{\lambda}} SE_1(e_w(\cdot) = 0) \oplus ker\left(Je_w(\bar{\lambda})\right) = T_{\bar{\lambda}}\triangle^2$

b) $T_{\bar{\lambda}} SE_1(e_w(\cdot) = 0) = ker\left(Je_w(\bar{\lambda})\right)$.

If (a) occurs then one can choose a system of coordinates (λ_1, λ_2) centered at $\bar{\lambda}$ and (y_1, y_2) centered at 0 such that e_w is the map:

$$\begin{cases} y_1 &=& \lambda_1 \\ y_2 &=& \lambda_2^2 \end{cases}$$

This is a special case of the submersion with folds. Recall that near a regular economy, the number of elements of $\lambda \in \mathcal{E}_\infty$ are constant, and the set of singular economies is exactly the set of economies in which the number of equilibria is not locally constant. This set is called envelope, see [Thom, R. (1962)].

In case (b) $\bar{\lambda}$ is a cusp. In this case can find coordinates (λ_1, λ_2) centered at $\bar{\lambda}$ and (y_1, y_2) centered at 0 such that e_w is the map:

$$\begin{cases} y_1 &=& \lambda_1 \\ y_2 &=& \lambda_1\lambda_2 + \lambda_2^3 \end{cases}$$

The proof of these claims follow as straightforward applications of theorems 2.2 , 2.4 and 2.5 of chapter III in [Golubistki, M. and Guillemin,V.(1973)].

We conclude that the singularities in an economic model, where e_w is a smooth mapping between 2-manifold, are folds or cusps.

Definition 4 *For each economy w such that e_w is a one generic map, we define $SE_{k,s}(e_w)$ as the set of points where the jacobian of the restricted map $e_w|SE_k(e_w(\cdot) = 0)$ drops rank s and let $SE_{k,s}(e) = \cup_w SE_{k,s}(e_w)$.*

Recall that as $SE_k(e_w(\cdot) = 0)$ is a manifold, then we can ask about the characteristics that exhibit e_w restricted to $SE_k(e_w(\cdot) = 0)$. In the theory of singularities it is proved that the singularities of this restricted map are unstable, in the sense that if the function e_w is slightly perturbed it exhibit a different qualitative character, for instance new critical point appear in the neighborhood of the original initial point, thereby these singular economies, w with degenerates λ in equilibrium will be called catastrophe set. Perturbation in the function e_w may appear from slightly changes in the utility functions or from a reallocation of the endowments.

Example 3 *In an economic model with $n = 3$ the points $SE_{1,0}(e)$ are folds, and the points $SE_{1,1}(e)$ are cusp, and there is not another kind of singularities..*

Theorem 4.1 of the chapter VII in [Golubistki, M. and Guillemin,V.(1973)] show the possibility to obtain a generic characterization of the set of the singularities $S_k(f)$ where a one generic map f restricted to this set, drops rank s. The set is noted $S_{s,k}(f)$.

The theorem above allow us to give the following characterization for this kind of singular economies.

If je_wf is transversal to $SE_{k,s}(e_w)$ and $\bar{\lambda} \in SE_k(e_w(\cdot) = 0)$ then there exist a coordinates system centered at $\bar{\lambda}$ and a coordinates system $y_1, y_2, ..., y_n$ centered at $e_w(\bar{\lambda})$ such that e_w has the form :

$$e_w(\lambda_1, \lambda_2, ...\lambda_n) = (h(\lambda), \lambda_2, ...\lambda_n)$$

$$h(\lambda) = \lambda_2\lambda_1 + \ldots + \lambda_k\lambda_1^{k-1} + \lambda_1^{k+1}.$$

The two examples below are a straightforward application of the later statement.

Example 4 *The swallows tail form may appear in economic models with 5 or more agents.*

To prove this claim consider $e_w : \triangle^4 \rightarrow \Re^4$ by the last statement, there exist a coordinate system $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ centered at critical $\bar{\lambda}$ and a coordinates system in $\Re^4$, centered at 0, such that the

$$e(\lambda) = (h(\lambda), \lambda_2, \lambda_3, \lambda_4)$$

$$h(\lambda) = \lambda_1^5 + \lambda_1^3\lambda_2 + \lambda_1^2\lambda_3 + \lambda_1\lambda_4.$$

Example 5 *In a model with 4 consumers for a neighborhood of a singular economy, $\bar{w}$ has a generic singularity $SE_{1,3}(e_{\bar{w}})$, definite by the map:*

$$\begin{cases} y_1 &= \lambda_1^4 + \lambda_1^2\lambda_3 + \lambda_1\lambda_2 \\ y_2 &= \lambda_2 \\ y_3 &= \lambda_3 \end{cases}$$

6 Dynamics on the Equilibrium Manifold of an Inter-temporal Perfect Foresight Model

We consider a economy with finite horizon T, with n agents, and l goods in each $t \in [0, T]$. Each agent has a utility function :

$$U_i(x) = \int_{[0,T]} u_i(x(t))d\mu(t), \ i \in \{1, 2 \ldots, n\}.$$

Where $\{S = [0,T], \mathcal{F}, \mu\}$ is a measure space, $x : [0,T] \to R^l$ is a vectorial field of $L^q(\mu), 2 \leq q < \infty$, measurable functions on [0, T] into R^l, and for each agent u_i is a C^2 and strictly concave function, such that $lim_{x_j(t) \to 0} \frac{\partial u_i(x(t))}{\partial x_j} = \infty$ for all $i \in \{1, 2, \ldots n\}$ and $j \in \{1, 2, \ldots l\}$. Each agent has a measurable set of endowments,$w_i \in (L^q(\mu))^l$ such that $w_i : [0,T] \to R^l$, and has a perfect foresight; that is, he knows both current and future values of w_i, and take them as given. In these condition the Negishi approach follows as in [Accinelli, E. (1996)].

The agent's decision problem is described as the simultaneous choice of an allocation in the current period and a plan for the future, constrained to lie in his budget set. This simultaneous decision will be represented by a measurable vector field x_i, on $L^q(\mu), 2 \leq q < \infty$ into R^l constrained to lie in his budget constraint, such that maximize his utility function.

Let $w = \{w_1, w_2, \ldots w_n\}$ be an endowment process, each w_i is a vector field of l measurable functions of $L^q(\mu)$. From section 3 we have that for each w, there exists a finite set of $\lambda \in \Delta^{n-1}$, such that $e(\lambda, w) = 0$.

For each λ there exists a system of equilibrium prices $P(t)$, and the respective allocation $x(t)$ of equilibrium. The number of λ for each w in the equilibrium manifold is odd.i The set of singular economies is exactly the set of economies for which the number of equilibria is not locally constant.

Suppose now that the inter-temporal agent maximizes from $t = 0$ for all $t \in [0,T]$, as he knows his function w each agent knows his potential paths of equilibrium. That is, from $t = 0$ each agent knows the future prices and the future allocations of the equilibrium.

Let us consider a model with singular economies. For a w fixed, we obtain a multivoque function $\Lambda : [0,T] \to \Delta^{n-1}$, such that $\Lambda(t) = \{\lambda \in R^n : e(\lambda, w(t)) = 0\}$. In this way, the evolution of economic system is then viewed as a sequence of competitive equilibrium in each point of time.

Let $\gamma : [0,T] \to \Delta^{n-1}$ a selection that satisfies $\gamma(t) \in \Lambda(t)$ each $\gamma(t)$ is a vector field in Δ^{n-1}.

Then for a given endowments process w we can define the equilibrium in terms of a welfare weights paths as follows.

Definition 5 *A selection γ, such that $\gamma(t) \in \Lambda(t)$ is an equilibrium selection if $e(\gamma, w) = 0$.*

A singular economy $\bar{w}$ is a bifurcation point, going through $\bar{w}$ the number of branches of equilibrium increase by two. Then there exist a regular economy with at least three equilibrium welfare weights, $\lambda_1, \lambda_2, \lambda_3$.

Let us now consider the following multivoque function:

$$\Lambda(t) = \begin{cases} = \lambda_0 \in \triangle^{n-1} & t < \bar{t} \\ = \{\lambda_1, \lambda_2, \lambda_3\} & t > \bar{t}, \quad \lambda_k \in \triangle^{n-1} \end{cases},$$

where $e(\lambda_1, w) = 0$, $0 \le t < \bar{t}$, and for each $k = 1, 2, 3$; $e(\lambda_k, w) = 0$, $\bar{t} \le t \le T$.

Consider now the following partition $\mathcal{T}$ of $[0, T]$:

$$\mathcal{T} = \{[0, t_1], (t_1, t_2)(t_2, t_3)(t_3, T]\}, \ t_1 = \bar{t}.$$

Let us now define the following sequence of equilibrium selections:

$$\gamma_1(t) = \begin{cases} \lambda_0 & 0 \le t \le t_1 \\ \lambda_k & t \in (t_k, t_{k+1}) \ k = \{1, 2, 3\} \end{cases}$$

Let $(t_{11}, t_{12})(t_{12}, t_{13})(t_{13}, t_{14})$ be a partition of $(t_1, t_2), t_1 = t_{11}; t_2 = t_{14}$.

$$\gamma_{12}(t) = \begin{cases} f_1(t) & t \notin (t_1, t_2) \\ \lambda_i & t \in (t_{1i}, t_{1(i+1)}) \ i = \{1, 2, 3\} \end{cases}$$

Let $\{\gamma_{k_1 k_2 \dots k_n}\}$ $k \in \{1, 2, 3\}, m \in \{1, 2, \dots\}$ be the sequence of selections built in this way.

Let P_k be the unique equilibrium system of prices for λ_k, $i = 1, 2, 3$, from this correspondence we can find, for each selection in the sequence, an unique equilibrium price system $P_{k_1 k_2 \dots k_m} \in L^p(\mu), k \in \{1, 2, 3\}$ and $m \in \{1, 2, \dots\}$, such that for a given $\epsilon > 0$, $\|P_{k_1 k_2 \dots k_m} - P_{k_1 k_2 \dots k_{m+r}}\|_p < \epsilon$, for all $m > m_0$ and $r > 0$.

Then we had proved the following theorem:

Theorem 7 *Inter-temporal perfect foresight model, with separable, C^2 and strictly concave utility function, with L^q as consumption space, and with singular economies, have infinite number of pairs (x, P) of equilibrium paths and this set is not isolate.*

Observe that there exists the possibility to obtain a chaotic path of equilibrium, because the system of equilibrium prices P may has a infinite number of jump across the branches in the equilibrium manifold.

7 Final Comments

The follow statements are comments about the infinite dimensional economies.

- In this paper we introduce the excess utility function showing that it is a powerful tool in order to characterize the equilibrium set. In this sense, the excess utility function appears

as a good substitute in infinite dimensional economies, for the generally inexistent, excess demand function. However this similar mathematical form, the excess utility function and the excess demand function do not have the same economic interpretation.

- The main object of this work is to characterize the singular submanifold of equilibria (SE). The emphasis on singular economies is the specific difference to the recent papers of [Balasko, Y. 1997a] and [Balasko, Y. 1997b]. However the relatively small size in mathematical terms of the singular economies in the equilibrium manifold, has a significant importance from an economic point of view, principally if the object of the economic analysis is the change in the structure of the endowment distribution.

- In section five we show that the modern theory of singularities is very relevant for the classification of singular economies on the equilibrium manifold in infinite dimensional models. Similar proves can be done for finite dimensional models. In this cases the excess demand function would be a good tool to analice the set of singular economies.

- Finally we would like to indicate that the future may be an absolutely unpredictable path inside a deterministic model. The possibility of chaos emerge in a very predictable model as our inter-temporal perfect foresight model set in the last section.

In order to continue with other studies about this subject, we think that the singular economies should be analyzed with the methods of the " Bifurcation Theory", see for instance [Castrigiano, P. L. D.; Hayes, S. A.].

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