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Abstract

The aim of this paper is to characterize the set of non regular economies, when there is a finite number of consumers with either finite or infinitely many goods. We prove that the structure of the equilibrium set is the same in both cases and that there exist bifurcations in some of these models.

Employing results on singularities, for economies with two goods and three or four consumers, we obtain a new result in uniqueness of the equilibrium, and we prove that the set of critical economies is a closed subset of zero measure, in the equilibrium manifold.

1 Introduction

A large part of the results of General Equilibrium Theory can be summarized by saying that the equilibrium set does not show qualitative changes as long as the initial endowments vary within the same connected component of the set of regular economies. Discontinuities in the number of equilibria can be observed only when initial endowments are varied in such a way that they come across singular economies. Improve in our understanding of the nature of these discontinuities requires an investigation of the singularities of the natural projection.

In this paper the manifold structure of the equilibrium set plays a prominent role.

It is well known that the demand function is a good issue to deal with the equilibrium manifold in economies in which consumption spaces as subset of R^n [Mas-Colell, A. (1985)], but unfortunately if the consumption spaces are subsets of infinite dimensional spaces, the demand function may not exist, [Araujo, A. (1987)]. However it is possible to characterize the equilibrium manifold from the excess utility function, see for instance [Accinelli, E. (1995)].

We will consider that the hypotheses that assure the existence either of the demand function or of the excess utility function are satisfied, there are n consumers their consumption space defined on R^l, or there are l goods in each state of the world in the case of infinitely many goods.

Following [Balasko, Y. (1988)], we generalize his results in at least two senses. First, using the excess utility function, we prove that the equilibrium manifold, for economies with infinite many goods, has the same properties that the equilibrium manifold for economies with a finite quantity of goods. In the other hand, we obtain new results on uniqueness, employing the modern clasification of singularities.

2 Definitions and the Model

Let $\mathcal{E} = \{(p, w) \in S^{l-1} \times \Omega : z(p, w) = 0\}$ be the equilibrium manifold, where $z : S^{l-1} \times \Omega \to R^{l-1}$ is the excess demand function, $S^{l-1} = \{p \in R^l; \sum_{i=1}^{l} p_i^2 = 1\}$ and $\Omega = R^{ln}$.

To prove that, in the infinitely many goods case the equilibrium set is a manifold, we will use a standard application of the Transversality Theorem.

Definition 1 *Let $f : X \to Y$ be a mapping between two Banach spaces, X and Y. The point $x \in X$ is called a regular point of f if it is a submersion at x, i.e. f is a C^1 mapping in a neighborhood of x, if the F-differential, $Df(x) : X \to Y$ is surjective and if the null space $N(Df(x))$ splits X, see [Zeidler, E. (1993)]. Otherwise x is called singular.*

Definition 2 *Let $f : D(f) \subset X \to Y$ be a mapping between two Banach spaces, X and Y over R, f is called a submersion at the point x if f is a C^1 mapping on a neighborhood of x, if $f'(x)$ $X \to Y$ is surjective and if the null space $N(f'(x))$ splits X.*

Denote the consumption space by $\Omega_\infty = \Pi_{i=1}^n \mathcal{A}_i$, where $\mathcal{A}_i$ is the consumption space for each consumer, $i = 1, 2, \ldots, n$. We make $\mathcal{A}_i$ in a Banach Space.

Theorem 1 *Let M be the set of endowments w, such that, 0 is a regular value for the excess utility function $e(\lambda, w)$ with $\lambda \in \Delta^{n-1}$, the $n-1$ dimensional simplex. Then:*

1) M is an open and dense set in Ω_∞.

2) The equilibrium set, $\mathcal{E}_\infty = \{(\lambda, w) \in \Delta^{n-1} \times \mathcal{M} : e(\lambda, w) = 0.\}$ is a nl dimensional manifold.

Proof: Consider an economy w and the following redistribution of the initial endowments:

Let $h = (h_1, \ldots, h_n) \in \Omega_\infty$, and $t = (t_1, t_2, \ldots, t_n) \in R^n$, where $t_n h_n(s) = -\sum_{i=1}^{n-1} t_i h_i(s)$; define $e(\lambda, w + th)$ by:

$e(\lambda, w + th) =$

$$= \{\int_S \gamma(s,\lambda)(x_1(s,\lambda) - w_1(s) - t_1 h_1(s))d\mu(s), \ldots, \int_S \gamma(s,\lambda)(x_n(s,\lambda) - w_n(s) - t_n h_n(s))d\mu(s)\}. \tag{1}$$

From the Transversality Theorem, [Arnold, V.; Varchenko, A.; Goussein-Zadé, S.], (ch. 1,2) there exists $t \in R^n$, arbitrarily close to 0, such that the economy $w + th$, is no singular. Then the set of economies with 0 as regular value of $e(\lambda, w)$ is open and dense.

It is obvious that $rank J_t e(\lambda, w + th) = n - 1$. It follows that $\mathcal{E}_\infty$, is a manifold of dimension nl.[]

Following [Balasko, Y. (1988)], we will characterize each economy by its endowments $w \in \Omega$, (preferences are fixed) and we will say that an economy is regular if $\Pi^{-1}(w)$ has no critical points, where $\Pi : S^{l-1} \times \Omega \to \Omega$ is the natural projection.

In the same form we will say an economy is regular if $\Pi_\infty^{-1}(w)$ has not critical points, where $\Pi_\infty \Delta^{l-1} \times \Omega_\infty \to \Omega_\infty$ is the natural projection.

Note, the first characterization is for economies with finite number of goods, and the second one for economies with infinitely many goods. In each of these two cases we will say that an economy is singular if is not a regular one.

3 Singularities on the Equilibrium Manifold

When w defines a regular economy, an infinitesimal variation of the paramenter w, entails an infinitesimal variation of the corresponding equilibrium prices. This stabilty property is true in a residual set of Ω. This property explains the relative constancy of equilibrium prices through time. On the other hand, those sudden changes from one equilibrium level to another which are observed from time to time may correspond to initial endowments crossing singular economies. This structural stability should not be confused with stability in the sense of tatonnement.

Let us recall that a singular value is just the image of a critical point, the set of regular values is by definition the complement of the set of singular values. One of the most useful theorem in differential topology is Sard's theorem which describes the set of singular values of a smooth mapping. This theorem states that the set of regular values of f is open and dense in the image of f, see [Milnor, J. (1965)]. For the infinite dimensional case the so called Sard-Smale theorem generalizes the Sard's theorem to Banach spaces. [Zeidler, E. (1993)]..

Remark 1 *It is well known that if w is a regular economy there are an odd number of equilibrium prices, that is generically for each w the set $P = \{p \in S^{l-1} : z(\cdot, w) = 0\}$ has an odd number of elements. [Mas-Colell, A. (1985)]. Analogously for e, the excess utility function, generically in w there exists an odd number of λ such that $e(\lambda, w) = 0$. [Accinelli, E. (1995)].*

Balasko proves that, the set of singular economies has Lebesgue measure zero. See for example [Balasko, Y. (1988)]. Nevertheless this singular set play a very special role, when the endowments w crosss these economies each equilibrium leads to several new equilibrium forms (the bifurcation case). In this points we obtain discontinuity in prices or in social weights.

This apparently inexplicable and impredictable discontinuity leads to the serious, sometimes heated question of the market mechanism, and even to irrational behavior that occasionally ends in widespread destruction of resources through futile attempts to get back to the former price levels.

Suppose now that the economy has fixed total resources. Let $r \in R^l$ represent the vector of fixed resources, i.e., $\sum_{i=1}^{n} w_i = r$. Following [Balasko, Y. (1988)] we denote by $\Omega(r) = \{w = (w_1, \dots, w_n) \in (R^l)^n, \sum_{i=1}^{n} w_i = r\}$ the space of economies. For economies with infinitely many goods, we suppose that there exists in each state of the world $s \in \mathcal{S}$, the set $\Omega_\infty(r(s)) = \{w = (w_1, \dots w_n) : \sum_{i=1}^{n} w_i(s) = r(s)\}$ where r is a fixed vector in the Banach Space.

Let $\Pi : \mathcal{E}(r) \to \Omega(r)$, be the natural projection for economies with fixed resources, analogously let $\Pi_\infty : \mathcal{E}_\infty(r) \to \Omega_(\infty)$ be the natural projection for economies in infinitely dimensional Banach Space.

For the finite dimensional case, an economy w is singular if the Jacobian of Π has not full dimension. For arbitrary case, w is singular if Π_∞ is not a submersion.

Theorem 2 *The natural projection is onto.*

Proof: Let us assume the contrary, then there exists w either in $\Omega(r)$ or in $\Omega_\infty(r)$ such that either $\Pi^{-1}(w)$ or $\Pi_\infty^{-1}(w)$ is empty, then w is a regular value, then it has asociated an odd number of λ such that (λ, w) are in the equilibrium manifold. From this contradiction the theorem follows.[]

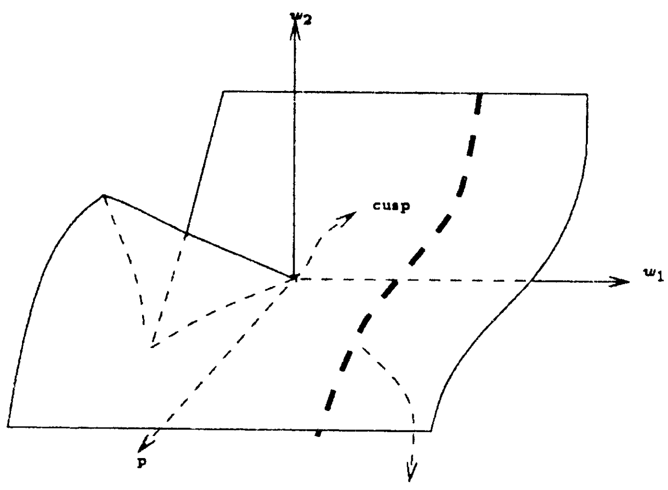
Suppose now that we are in a two goods two agents economy. In this case $\Pi : \mathcal{E}(r) \rightarrow \mathcal{R}^2$ where $\mathcal{E}(r) = \{(p, w) \in S^1 \times \mathcal{R}^2, z(p, w) = 0\}$. From [Golubistki, M. and Guillemin,V.(1973)] we know that if X and Y are 2-manifold, and f is a generic mapping between X and Y, one of the following two situations can occur:

a) $T_pS_1(f) \oplus ker(df)_p = T_pX$

b) $T_pS_1(f) = ker(df)_p$.

Where $S_1(f)$ is the set of singularities of f with codimension 1 in X.

For two goods, two agents economies with fixed resources, E, and Ω are 2-dimensional manifolds. Then Π is in the above condition. From the remark 1, we will consider only case b). In this case, the singularity has a simple cusp form, see fig. 1.



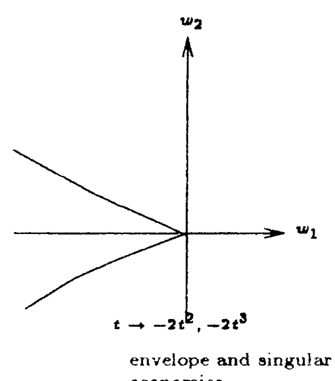


Figure 1: The Simple Cusp

The image of the singularities from the natural projection is called envelope [Thom, R. (1962)].

Let $W : \Omega(r) \rightarrow S^{l-1}$ be the equilibrium correspondence, i.e, $W(w) = \{p \in S^{l-1} : (p, w)$ is an equilibrium $\}$. It is well known, that if w is a regular economy then the number of elements of the set $W(w)$ is constant in a neighborhood of w, [Balasko, Y. (1988)].

The set of singular economies is exactly the set of economies for which the number of equilibria is not locally constant, see fig. 2.

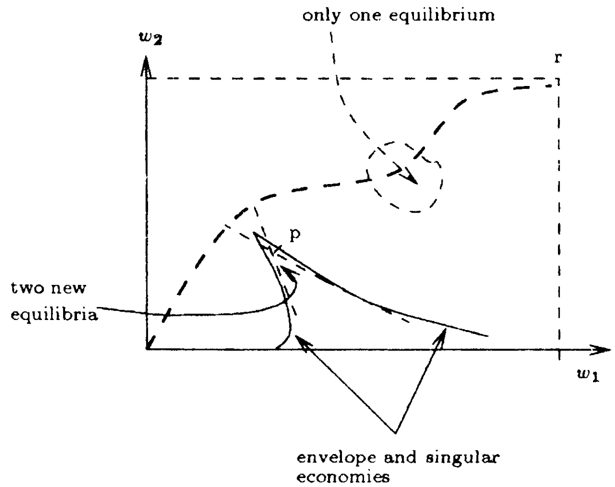


Figure 2: The Edgeworth Box

From Thom's clasification of the singularities and remark 1, follows that, the economies with two goods and three or four consumers, have not singularities. This implies that these economies have global uniqueness of equilibrium.

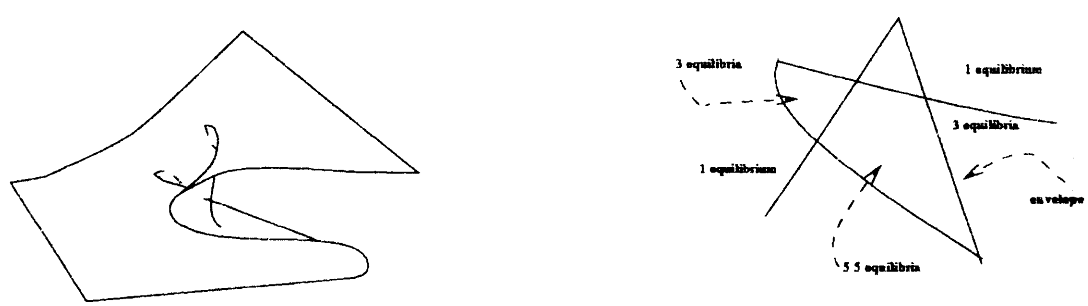


Figure 3: The Butterfly

For a two goods five consumer economies the singularities have the Butterfly form, see fig. 3.

In general we have the following characterization for the singularities :

Theorem 3 *If $X \to Y$ satisfies the transversality condition, $J^k S_{1_k}$ and $x_0 \in S_{1_k}(f)$, then there exists a coordinate system $x_1, \ldots, x_n$ centered at x_0 and a coordinate system $\{y_1, \ldots, y_n\}$, centered at $f(x_0)$ such that f has the form*

$$\begin{array}{rcl} f^*y_1 & = & x_2x_1 + \ldots + x_kx_1^{k-1} + x_1^{k+1} \\ f^*y_2 & = & x_2 \\ \vdots & \vdots & \vdots \\ f^*y_n & = & x_n \end{array}$$

[Golubistki, M. and Guillemin,V.(1973)], (Theorem 41, pag 177).

4 Singularities on Infinite Dimensional Economies

In this section, we prove that the natural projection, and the excess utility function, have the same singularities. That is, if Π_∞ is not a submersion, then the $rank J_\lambda e(\lambda, w) < n-1$.

Theorem 4 *The equilibrium (λ, w) is critical if and only if $rank J_\lambda e(\lambda, w) < n-1$.*

Proof: Recall that $e_i(\cdot,\cdot) : \Delta^{n-1} \times \Omega_\infty \to R^n, i = 1, \ldots, n$, and that

$$e(\lambda, w) = \{e_1(\lambda, w_1), \ldots, e_n(\lambda, w_n)\}.$$

We define the function F from $\Delta^{n-1} \times \Omega_\infty \to R^{n(l+1)}$, by associating with the generic element (λ, w) of $\Delta^{n-1} \times \Omega_\infty$ the value $F(\lambda, w) = (\lambda_1, \lambda_2, \ldots \lambda_n, w_1^1, w_1^2, \ldots \bar{w}_n^l)$ where: $\bar{w}_i^l = e_i(\lambda, w_i) + w_i^l$, for all $i = 1, 2 \ldots n$.

In this coordinate system, if $(\lambda, w) \in \mathcal{E}_\infty$ then, $\bar{w}_i^l = w_i^l$. And for $\Pi_\infty : \mathcal{E}_\infty \to \Omega_\infty$, we obtain the following identity:

$$\begin{array}{l} w_1^1 = w_1^1 \\ \vdots \\ w_1^{l-1} = w_1^{l-1} \\ w_1^l = e_1(\lambda, w_1) + w_1^l \\ \vdots \\ w_n^1 = w_n^1 \\ \vdots \\ w_n^{l-1} = w_n^{l-1} \\ \bar{w}_n^l = e_n(\lambda, w_n) + w_n^l \end{array}$$

From Π_∞ we obtain the Jacobian Matrix:

$$J\Pi_\infty(\lambda, w) = \begin{bmatrix} \mathbf{O} & \mathbf{B} \\ J_\lambda e(\lambda, w) & \mathbf{C} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

$\mathbf{B}$ is a $n(l-1) \times n(l-1)$ matrix.[]

The set of pairs $(\lambda,\omega) \in \mathrm{E}_\infty$ such that the rank of the matrix $J_\lambda e(\lambda,\omega)$ is less than the maximal, $n-1$ consists of the singularities of the mapping e, or equivalently the set of singular economies.

Let $k = \{0, 1, \ldots, (n-2)\}$, define

$$S_k(e) = \{(\lambda,\omega) \in \mathcal{E}_\infty : rank J_\lambda e(\lambda,\omega) = k, \}$$

and let $S(e) = \cup_{k=0}^{n-2} S_k(e)$. The set $S(e)$ consists of the singular economies.

The rank of the mapping $J_\lambda e(\lambda,\omega)$ is defined to be the dimension of its image.

Let $\mathcal{L}(n-1)$ denote the space of linear maps from $R^{n-1} \to R^{n-1}$, so in for all (λ,ω) we have that, $J_\lambda e(\lambda,\omega) \in \mathcal{L}(n-1)$. For any $k = \{0, 1, \ldots, n\}$ let $\mathcal{L}_k(n-1)$ be the elements of $\mathcal{L}(n-1)$ of rank k. Let $J_\lambda : \mathcal{E}_\infty \to \mathcal{L}(n-1)$ be the mapping which assigns to every point $(\lambda,\omega) \in \mathcal{E}_\infty$ the linear map $J_\lambda e(\lambda,\omega)$. Then we have that the point $(\lambda,\omega) \in \mathcal{E}_\infty$ is an element of the set $S_k(e)$ if and only if $J_\lambda e(\lambda,\omega) \in \mathcal{L}_k(n-1)$, or equivalently $S_k(e) = (J(e))^{-1}\mathcal{L}_k(n-1)$. That is an economy is singular if it is an element of $\mathcal{E}_\infty$, such that the map $J_\lambda(e)$ intersects the set $\mathcal{L}_k(n-1)$ for $k \in \{0, 1, \ldots, (n-2)\}$.

Note that the codimension of $\mathcal{L}_k(n-1)$ in $\mathcal{L}(n-1)$ is $(n-1-k)^2$, and hence its dimension is $(n-1)^2 - (n-1-k)^2$. Then if $J_\lambda(e)$ is transversal to $\mathcal{L}_k(n-1)$, the set of singularities $S_k(e)$ will be empty or a manifold in $\mathcal{E}_\infty$ of codimension $(n-1-k)^2$, see [Golubistki, M. and Guillemin,V.(1973)] pag. 52.

By Thom's Transversality Theorem, [Golubistki, M. and Guillemin,V.(1973)] pag 54, we have that the set $T_k = \{e : J_\lambda(e)$ transversal to $\mathcal{L}_k(n-1)\}$, is a residual subset of $C^\infty(\mathcal{E}_\infty, \mathcal{L}_k(n-1))$. The residual is a subset which contains a countable intersection of open dense set.

We have the following theorem:

Theorem 5 *The set of singular economies with n agents and l goods in $S_k(e)$ is generically empty if:*

$$k > (n-1) + \sqrt{nl}. \tag{2}$$

Proof: From the fact that the codimension of $S_k(e)$ in $\mathcal{E}_\infty$, is equal to the codimension of $\mathcal{L}_k(n-1)$, in $\mathcal{L}(n-1)$, it follows that the dimension of the $S_k(e)$ is equal to $nl - (n-1-k)^2$. Then if $nl - (n-1-k)^2 < 0$, that is if $k > (n-1) + \sqrt{nl}$, $S_k(e)$ is empty. []

Example 1 *Suppose that the economy has 2 goods and 2 agents, then there are not singularities which $k \geq 3$.*

Substituing in eq. (2) the result follows.

But if $nl \geq 4$ it can be singularities of codimension 4.

Theorem 6 *The set of critical equilibrium is a closed subset of zero measure, of the equilibrium manifold.*

Proof It follows from the fact that every stratum $S_k(e)$ is a submanifold with strictly positive codimension, therefore it has measure zero.

As $S(e)$ is a finite union of sets of measure zero, itself has measure zero. []

References

[Accinelli, E. (1995)] "Notes about Uniqueness of Equilibrium on Economies with Infinitely Many Goods". *Série F-081-Maio/95, Informes de Matemática, IMPA.*

[Araujo, A. (1987)] "The Non-Existence of Smooth Demand in General Banach spaces". *Journal of Mathematical Economics* **17**, 1-11

[Arnold, V.; Varchenko, A.; Goussein-Zadé, S.] "Singularités des Applications Différeentiables" *Editions Mir, 1982.*

[Balasko, Y. (1988)] "Foundations of the Theory of General Equilibrium". *Academic Press, inc.*

[Golubistki, M. and Guillemin,V.(1973)] "Stable Mappings and Their Singularities". *Springer Verlag.*

[Mas-Colell, A. (1985)] "The Theory of General Equilibrium, A Differentiable Approach". Cambridge University Press.

[Mas-Colell, A. (1990)] "Indeterminaci in Incomplete Market Economies." *Economic Theory* **1**

[Milnor, J. (1965)] "Topology from the Differential Viewpoint". *University of Virginia Press: Charlottesville.*

[Thom, R. (1962)] "Sur la Théorie des enveloppes".*Journal de Mathematiques,* XLI.

[Zeidler, E. (1993)] "Non Linear Functional Analysis and its Applications"**(1).** *Springer Verlag.*