

Asymptotic $U(1)$ charges at spatial infinity

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ABSTRACT: Large gauge symmetries in Minkowski spacetime are often studied in two distinct regimes: either at asymptotic (past or future) times or at spatial infinity. By working in harmonic gauge, we provide a unified description of large gauge symmetries (and their associated charges) that applies to both regimes. At spatial infinity the charges are conserved and interpolate between those defined at the asymptotic past and future. This explains the equality of asymptotic past and future charges, as recently proposed in connection with Weinberg's soft photon theorem.

KEYWORDS: Gauge Symmetry, Global Symmetries

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1 Introduction

The subject of asymptotic symmetries in gravity and gauge theories has experienced renewed interest after the discovered [1–5] relation with so-called soft theorems [6]. In general relativity, the occurrence of novel asymptotic symmetries originally appeared in the study of gravitational waves [7, 8] whereas in electromagnetism, large U(1) symmetries were

studied in relation to infrared divergences in QED [9]. There is in fact a tight connection between asymptotic symmetries and infrared issues in both gravity and electromagnetism as emphasized long ago [10] and more recently [11–14].

The key insight of Strominger and collaborators was the realization that the well-established soft theorems can be understood as the existence of infinitely many conserved charges associated to asymptotic symmetries. Here we would like to better understand such charges and their conservation from a classical field theory perspective. In the case of electromagnetism, the conservation statement associated to the leading soft photon theorem can be written as the following “continuity” condition of the field strength at future and past null infinities [15]:

$$\overset{-2}{F}_{ru}(u = -\infty, \hat{x}) = \overset{-2}{F}_{rv}(v = \infty, -\hat{x}). \tag{1.1}$$

Here $\overset{-2}{F}_{ru}(u, \hat{x})$ is the leading term in an $r \rightarrow \infty$ expansion of the field strength at future null infinity and $\overset{-2}{F}_{rv}$ the analogue term at past null infinity.¹

Our first goal will be to show how condition (1.1) can be understood as a consequence of Maxwell equations near spatial infinity. Here we will follow the idea [21–23] of using the space-infinity hyperboloid as a way to link future and past null infinities.

We will then study large U(1) charges from the perspective of spatial infinity. As reviewed below, the smeared version of the quantities (1.1) have the interpretation of generators of large U(1) gauge transformations at future and past infinities. We will use the expansion of the Maxwell field at spatial infinity [21, 22, 24, 25] to define large U(1) generators at *arbitrary times* on the space-infinity hyperboloid. The charges will be shown to be conserved and to interpolate between the future and past infinity charges. The end result will be a unified description of large U(1) symmetries and charges that applies to both spatial and future/past infinities. In this way, the present article extends the work [26] that provides a unified picture of null and time-infinities charges.

We would like to mention that our first result, namely deriving eq. (1.1) from the field equations (plus fall-off assumptions), is not new but was established by Herdgen long ago [27]. Herdgen’s techniques are however quite different from the ones presented here and we leave for the future a detailed comparison between the two approaches.²

¹The quantity $\overset{-2}{F}_{ru}(u, \hat{x})$ is also referred to as ‘charge aspect’ in analogy to Bondi’s ‘mass aspect’. The conservation statement associated to the leading soft graviton theorem is a continuity condition on Bondi’s mass aspect [1–5]. Similar continuity conditions are behind soft theorems for other spins [16–20].

²We take the opportunity to address Herdgen’s concerns [28] regarding the interpretation of the charges as generators of large gauge transformations. Herdgen works in Lorenz gauge and concludes there are no large residual gauge transformation. In our understanding, his conclusion may be reached as follows: in Lorenz gauge, residual gauge transformations are given by gauge parameters satisfying the wave equation $\square\Lambda = 0$. Regular solutions to the wave equation are known to decay as $1/r$ in null directions (see e.g. [29]). Hence large gauge transformations are forbidden. Indeed, the large gauge parameters considered here and in [26, 30] are singular at $r = 0$. In our view, the catch in the argument is that in the discussions of large gauge symmetries the gauge conditions are only being used *asymptotically*. One may envisage a modification of the Lorenz gauge condition in the interior so as to ensure a regular gauge parameter everywhere.

The outline of the paper is as follows. In section 2 we set up notation and describe the assumed asymptotic properties of the fields at null, spatial and time infinities. Some derivations in this section are given in appendix A. In section 3 we present the argument for eq. (3.1). The argument rests on properties of the field equations at spatial infinity that are described in appendix B. In section 4 we study large gauge symmetries at spatial infinity and obtain the associated charges. The analysis requires a careful handling of the covariant phase space which may be of interest in its own right. In section 5 we describe how dual magnetic charges may be constructed. In the final section we summarize our results and point out potential connections with other works.

2 Preliminaries

The system of study will be a Maxwell field A_a in four dimensional flat spacetime coupled to massive charge fields. For definitiveness we consider a single massive charged scalar φ but the discussion applies to any number and kind of massive particles. The field equations are

$$\nabla^b F_{ab} = j_a, \tag{2.1}$$

$$\mathcal{D}^a \mathcal{D}_a \phi - m^2 \phi = 0, \tag{2.2}$$

where ∇_a is the spacetime covariant derivative, $\mathcal{F}_{ab} = \partial_a A_b - \partial_b A_a$ the field strength,

$$j_a = ie\phi(\mathcal{D}_a \phi)^* + \text{c.c.}, \tag{2.3}$$

the charge current and \mathcal{D}_a the gauge covariant derivative, $\mathcal{D}_a \phi = \partial_a \phi - ieA_a \phi$.

Local U(1) gauge transformations are parametrized by a scalar Λ and act as

$$\delta_\Lambda A_a = \partial_a \Lambda, \quad \delta_\Lambda \phi = ie\Lambda \phi. \tag{2.4}$$

Large U(1) gauge parameter will be discussed in the context of Lorenz gauge

$$\nabla^a A_a = 0, \tag{2.5}$$

in which the *residual* gauge parameters Λ satisfy the wave equation

$$\nabla^a \nabla_a \Lambda = 0. \tag{2.6}$$

In the following subsections we describe expansions of the fields at null, space and time infinities. We will assume that in a neighborhood of each infinity the system is described by the free (linearized) theory. The scalar field being massive will only register at time-like infinity. The Maxwell field will have nontrivial components at all infinities.

Notation: the discussion involves many spaces and notation may be confusing at first. We here summarize the notation for coordinates, metric and derivative for each space:

- spacetime \mathbb{R}^4 : x^a, η_{ab}, ∇_a
- sphere S^2 : x^A, q_{AB}, D_A

- spatial infinity \mathcal{H}^o : $y^\alpha = (\tau, x^A), h_{\alpha\beta}, D_\alpha$
- future time-like infinity \mathcal{H}^+ : $y^\alpha = (\rho, x^A), h_{\alpha\beta}, D_\alpha$
- future null infinity \mathcal{I}^+ : $(u, x^A), q_{AB}, (\partial_u, D_A)$

\hat{x} denotes the unit three-vector determine by a sphere point x^A . Functions f on the sphere are denoted as $f(\hat{x})$.

We warn the reader that the coordinates ρ and τ have different meaning depending on whether we are discussing expansions at space or at time infinities.

2.1 Field expansion at null infinity

The (future) null infinity regime can be studied in outgoing coordinates in terms of which the Minkowski line element takes the form

$$ds^2 = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B. \tag{2.7}$$

Here $r = \sqrt{x^2 + y^2 + z^2}$ is the radial coordinate, $u = t - r$ the retarded time, and $x^A, A = 1, 2$ coordinates on the sphere. q_{AB} is the unit sphere metric. We use the notation \hat{x} to denote the unit three-vector parametrized by the point on the sphere x^A . For concreteness we focus the discussion on future null infinity. Similar considerations apply to past null infinity with $v = t + r$ playing the role of u .

Future null infinity is reached by taking the limit $r \rightarrow \infty$ with $u, \hat{x} = \text{const}$. The abstract manifold of such limiting endpoints is denoted by \mathcal{I}^+ . It has the topology of a cylinder, parameterized by (u, \hat{x}) .

We assume the Maxwell field satisfies the standard asymptotic expansion [1–5]:

$$F_{AB}(r, u, \hat{x}) = \sum_{n=0}^{\infty} r^{-n} \bar{F}_{AB}^{-n}(u, \hat{x}) \tag{2.8}$$

$$F_{ru}(r, u, \hat{x}) = \sum_{n=2}^{\infty} r^{-n} \bar{F}_{ru}^{-n}(u, \hat{x}), \tag{2.9}$$

and similar expansion for the remaining components, with $F_{Au} = O(r^0)$ and $F_{Ar} = O(r^{-2})$. Since the charged scalar field is massive, we assume $F_{\mu\nu}$ satisfies the free field equations near null infinity. To solve the equations it is convenient to fix a gauge and work with the vector potential A_μ . Either in radial gauge $A_r = 0$ or in Lorenz gauge one arrives at equations that can be recursively solved in terms of unconstrained, free data $\overset{0}{A}_A(u, \hat{x})$ (plus integration constants; see e.g. [31]). For instance, the leading $\nabla^b F_{ub} = 0$ equation yields:

$$\partial_u \bar{F}_{ru}^{-2} = D^A \overset{0}{F}_{Au}, \tag{2.10}$$

with $\overset{0}{F}_{Au} = -\partial_u \overset{0}{A}_A$. This can be solved in terms of the free data after fixing the integration ‘constant’

$$\bar{F}_{ru}^{-2}(u = -\infty, \hat{x}). \tag{2.11}$$

For the free data we assume standard $u \rightarrow \pm\infty$ fall-offs:

$${}^0 A_A(u, \hat{x}) = {}^0 A_A(\hat{x})^\pm + O(|u|^{-\epsilon}). \tag{2.12}$$

In appendix A.1 we show that the field equations together with ${}^0 A_A(u, \hat{x}) = O(1)$ imply

$${}^{-n} F_{ru}(u, \hat{x}) = O(|u|^{n-2}) \tag{2.13}$$

at $u \rightarrow \pm\infty$.

The gauge parameter Λ has a similar expansion with

$$\Lambda(r, u, \hat{x}) = \lambda_+(\hat{x}) + \sum_{n=1}^{\infty} \left(r^{-n} {}^{-n} \Lambda(u, \hat{x}) + r^{-n} \ln r {}^{-n, \ln} \Lambda(u, \hat{x}) \right). \tag{2.14}$$

The presence of logarithmic terms is due the $O(1)$ term [30].

2.2 Field expansion at spatial infinity

To study the fields at spatial infinity we use hyperbolic coordinates in the $r > |t|$ region,

$$\rho := \sqrt{r^2 - t^2}, \quad \tau := \frac{t}{\sqrt{r^2 - t^2}} \tag{2.15}$$

in terms of which the Minkowski line element reads:

$$ds^2 = d\rho^2 + \rho^2 d\sigma^2 \tag{2.16}$$

with

$$d\sigma^2 = -\frac{d\tau^2}{1 + \tau^2} + (1 + \tau^2) q_{AB} dx^A dx^B =: h_{\alpha\beta} dy^\alpha dy^\beta \tag{2.17}$$

the (unit-radius) three-dimensional de Sitter metric. We denote by \mathcal{H}^o the abstract unit de Sitter metric that represents spatial infinity. For later convenience we introduced the notation $y^\alpha = (\tau, x^A)$ for coordinates on \mathcal{H}^o .

Following [24], we assume the Maxwell field has a $\rho \rightarrow \infty$ expansion

$$F_{\alpha\beta}(\rho, y) = \sum_{n=0}^{\infty} \rho^{-n} {}^{-n} F_{\alpha\beta}(y) \tag{2.18}$$

$$F_{\alpha\rho}(\rho, y) = \sum_{n=1}^{\infty} \rho^{-n} {}^{-n} F_{\alpha\rho}(y). \tag{2.19}$$

Since the charged field is massive we assume $F_{\mu\nu}$ satisfies the free field equations near spatial infinity. These equations yield evolution equations for the family of fields ${}^{-n} F_{\alpha\beta}, {}^{-n} F_{\alpha\rho}$ on \mathcal{H}^o . Free field asymptotic conditions also determine the $\tau \rightarrow \pm\infty$ behavior of these functions. For instance (see appendix A.2):

$${}^{-n} F_{\tau\rho}(\tau, \hat{x}) = O(|\tau|^{n-4}), \tag{2.20}$$

a condition that will be used when relating with the fields at null infinity.

For the purposes of this paper we will only be interested in the leading equations:

$$D^\alpha \overset{-1}{F}_{\alpha\rho} = 0, \quad D_{[\alpha} \overset{-1}{F}_{\beta]\rho} = 0, \quad (2.21)$$

$$D^\beta \overset{0}{F}_{\alpha\beta} = 0, \quad D_{[\alpha} \overset{0}{F}_{\beta\gamma]} = 0, \quad (2.22)$$

where D_α is the covariant derivative on \mathcal{H}^o . Formally these equations look like electro- and magneto-static vacuum equations on \mathcal{H}^o , with $\overset{-1}{F}_{\alpha\rho}$ and $\overset{0}{F}_{\alpha\beta}$ playing the role of electric and magnetic fields respectively. They can be solved in terms of ‘potentials’ $\psi, \tilde{\psi}$ on \mathcal{H}^o as follows: The second equation in (2.21) implies that $\overset{-1}{F}_{\alpha\rho}$ is a total derivative:

$$\overset{-1}{F}_{\alpha\rho} = D_\alpha \psi \quad (2.23)$$

for some function ψ on \mathcal{H}^o defined up to an additive constant.³ The first equation in (2.21) then implies that ψ satisfies the wave equation on \mathcal{H}^o ,

$$D^\alpha D_\alpha \psi = 0. \quad (2.24)$$

Equations (2.22) take the same form as (2.21) when written in terms of the magnetic field $\frac{1}{2}\epsilon^{\alpha\beta\gamma}\overset{0}{F}_{\beta\gamma}$ where $\epsilon_{\alpha\beta\gamma}$ is the volume form on \mathcal{H}^o . They are then similarly solved in terms of a field $\tilde{\psi}$ such that

$$\overset{0}{F}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} D^\gamma \tilde{\psi}, \quad D^\alpha D_\alpha \tilde{\psi} = 0. \quad (2.25)$$

The gauge parameter has a similar $\rho \rightarrow \infty$ expansion

$$\Lambda(\rho, y) = \lambda(y) + O(\rho^{-1}). \quad (2.26)$$

Equation (2.6) then implies

$$D^\alpha D_\alpha \lambda = 0. \quad (2.27)$$

The $\tau \rightarrow \pm\infty$ behavior of λ is such that (see appendix B for further details)

$$\lambda(\tau, \hat{x}) = \lambda_\pm(\hat{x}) + O(\tau^{-\epsilon}). \quad (2.28)$$

By performing change of coordinates between (ρ, τ) to (r, u) coordinates, one can verify that $\lambda_+(\hat{x})$ in (2.28) coincides with $\lambda_+(\hat{x})$ in (2.14) (and similar statement for $\lambda_-(\hat{x})$ at past null infinity). See [26] for the analogous result at time-like infinity and appendix A.2 for analogous statement for the field strength.

2.3 Field expansion at time infinity

Coordinates adapted to time-like infinity are completely analogue to the space infinity coordinates with the role of radial and time coordinates interchanged. For concreteness we focus on future time-infinity. In the $r < |t|, t > 0$ region define:

$$\tau := \sqrt{t^2 - r^2}, \quad \rho := \frac{r}{\sqrt{t^2 - r^2}} \quad (2.29)$$

³If we work with a vector potential of the form $A_\alpha = O(\rho^0)$ and $A_\rho = O(\rho^{-1})$ then $\psi \equiv \overset{-1}{A}_\rho$. We will use this identification in section 4.

in terms of which the Minkowski line element reads:

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2 \quad (2.30)$$

with

$$d\sigma^2 = \frac{d\rho^2}{1 + \rho^2} + \rho^2 q_{AB} dx^A dx^B =: h_{\alpha\beta} dy^\alpha dy^\beta \quad (2.31)$$

the unit radius hyperbolic space. We denote by \mathcal{H}^+ this abstract hyperboloid, representing time-like infinity, with coordinates $y^\alpha = (\rho, x^A)$. Assuming free field $\tau \rightarrow \infty$ fall-offs for the Maxwell field and scalar one has:

$$F_{\alpha\beta}(\tau, y) = \sum_{n=0}^{\infty} \tau^{-n} \bar{F}_{\alpha\beta}^{-n}(y), \quad F_{\alpha\tau}(\tau, y) = \sum_{n=1}^{\infty} \tau^{-n} \bar{F}_{\alpha\rho}^{-n}(y) \quad (2.32)$$

$$j_\alpha(\tau, y) = \sum_{n=3}^{\infty} \tau^{-n} \bar{j}_\alpha^{-n}(y), \quad j_\tau(\tau, y) = \sum_{n=3}^{\infty} \tau^{-n} \bar{j}_\alpha^{-n}(y) \quad (2.33)$$

To leading order, the field equations are:

$$D^\alpha \bar{F}_{\tau\alpha}^{-1} = \bar{j}_\tau^{-3}, \quad D_{[\alpha} \bar{F}_{\beta]\tau}^{-1} = 0. \quad (2.34)$$

$$D^\beta \bar{F}_{\alpha\beta}^0 = 0, \quad D_{[\alpha} \bar{F}_{\beta\gamma]}^0 = 0 \quad (2.35)$$

As in the spatial infinity case, these equations can be solved in terms of scalars $\psi, \tilde{\psi}$ such that

$$\bar{F}_{\alpha\tau}^{-1} = \partial_\alpha \psi, \quad -D^\alpha D_\alpha \psi = \bar{j}_\tau^{-3} \quad (2.36)$$

$$\bar{F}_{\alpha\beta}^0 = \epsilon_{\alpha\beta\gamma} D^\gamma \tilde{\psi}, \quad D^\alpha D_\alpha \tilde{\psi} = 0. \quad (2.37)$$

They take precisely the form of electro and magneto static equations on \mathcal{H}^+ with a charge density \bar{j}_τ^{-3} . Solutions to these equations are described in appendix C.

For the gauge parameter we have

$$\Lambda(\tau, y) = \lambda(y) + O(\tau^{-1}). \quad (2.38)$$

with λ satisfying Laplace equation on \mathcal{H}^+

$$D^\alpha D_\alpha \lambda = 0. \quad (2.39)$$

3 \mathcal{I}^\pm conserved charges from spatial infinity perspective

In [15] it was argued that in a general scattering process of charged particles the following identity holds:

$$\bar{F}_{ru}^{-2}(u = -\infty, \hat{x}) = \bar{F}_{rv}^{-2}(v = \infty, -\hat{x}) \quad (3.1)$$

which represents an infinity of conservation laws between past and null infinity data. If we now smear (3.1) with sphere functions $\lambda_\pm(\hat{x})$ satisfying the ‘antipodal matching’ [1–5]

$$\lambda_+(\hat{x}) = \lambda_-(-\hat{x}), \quad (3.2)$$

and define:

$$Q_+[\lambda_+] := - \int_{S^2} d^2V \lambda_+(\hat{x}) \overset{-2}{F}_{ru}(u = -\infty, \hat{x}) \quad (3.3)$$

$$Q_-[\lambda_-] := - \int_{S^2} d^2V \lambda_-(\hat{x}) \overset{-2}{F}_{rv}(v = \infty, \hat{x}) \quad (3.4)$$

the conservation law reads [1–5]:

$$Q_+[\lambda_+] = Q_-[\lambda_-] \quad (3.5)$$

for all λ_{\pm} satisfying (3.2). The quantities $Q_{\pm}[\lambda_{\pm}]$ have the interpretation of large U(1) gauge charges at future/past infinities [15]: using eq. (2.10) one has,

$$Q_+[\lambda_+] = \int_{\mathcal{I}^+} dud^2V \lambda_+(\hat{x}) D^A \overset{0}{F}_{Au}(u, \hat{x}) - \int_{S^2} d^2V \lambda_+(\hat{x}) \overset{-2}{F}_{ru}(u = \infty, \hat{x}). \quad (3.6)$$

The first term in (3.6) is the ‘soft part’ of the charge and depends on the Maxwell field at null infinity. The second term in (3.6) is the ‘hard part’ of the charge which, upon using the field equations, depends on the massive field at time-like infinity. We refer the reader to appendix C for further details, where we show that the hard part (and hence $Q_+[\lambda_+]$) coincides with the expression obtained from covariant phase space methods [26]. Similar considerations apply to $Q_-[\lambda_-]$.

We now show that the conservation laws (3.1) are a consequence of the field equations at spatial infinity. The first task is to express each side of the equality (3.1) in terms of fields at spatial infinity. In appendix A.3 it is shown that the fall-offs (2.9), (2.13), (2.19), (2.20) imply:

$$\overset{-2}{F}_{ru}(u = -\infty, \hat{x}) = \lim_{\tau \rightarrow \infty} \tau^3 \overset{-1}{F}_{\rho\tau}(\tau, \hat{x}) \quad (3.7)$$

$$\overset{-2}{F}_{rv}(v = +\infty, \hat{x}) = - \lim_{\tau \rightarrow -\infty} \tau^3 \overset{-1}{F}_{\rho\tau}(\tau, \hat{x}). \quad (3.8)$$

Recall from eq. (2.23) that $\overset{-1}{F}_{\alpha\rho} = D_{\alpha}\psi$. From eq. (2.20) we then have that the $\tau \rightarrow \pm\infty$ asymptotic form of ψ is

$$\psi(\tau, \hat{x}) \xrightarrow{\tau \rightarrow \pm\infty} k_{\pm} + \tau^{-2}\psi_{\pm}(\hat{x}) + \dots \quad (3.9)$$

where k_{\pm} are possible $O(\tau^0)$ constants and the dots indicate subleading terms. Eqs. (3.7), (3.8) then take the form:

$$\overset{-2}{F}_{ru}(u = -\infty, \hat{x}) = 2\psi_+(\hat{x}) \quad (3.10)$$

$$\overset{-2}{F}_{rv}(v = +\infty, \hat{x}) = -2\psi_-(\hat{x}), \quad (3.11)$$

and the conservation law (3.1) translates into

$$\psi_+(\hat{x}) = -\psi_-(-\hat{x}). \quad (3.12)$$

This equality can now be established as a consequence of the wave equation (2.24) and the asymptotic form (3.9). A proof is given in appendix B.

4 Canonical charges at spatial infinity

The conserved charges $Q_{\pm}[\lambda_{\pm}]$ described in the previous section can be understood as large U(1) gauge charges at future/past infinities [15, 26]. Here we show that they can also be understood as large U(1) gauge charges at spatial infinity.

We first construct the covariant phase space symplectic structure associated to constant τ slices and consider large gauge U(1) symmetries at spatial infinity. We construct the associated canonical charges $Q_{\tau}[\lambda]$ and show that they are conserved. Their $\tau \rightarrow \pm\infty$ limit is then shown to reproduce the charges $Q_{\pm}[\lambda_{\pm}]$ of the previous section. In this way, the equality eq. (3.5) appears as a consequence of the conservation of $Q_{\tau}[\lambda]$.

During the analysis we will find interesting features in the asymptotics of the symplectic structure that may be of interest in their own right. We comment on them in a final subsection.

4.1 Symplectic current for $\tau = \text{const.}$ slicing

In the covariant phase space approach [32, 33], the symplectic product is defined as the flux through a Cauchy slice of a conserved symplectic current ω^a . For scalar QED, the standard current is given by:

$$\omega_{\text{bulk}}^a = \sqrt{\eta} \left(\delta F^{ab} \wedge \delta A_b + (\delta \mathcal{D}^a \phi)^* \wedge \delta \phi + \text{c.c.} \right) \quad (4.1)$$

(the wedge is in field space). The subscript ‘bulk’ is to indicate that the current may have to be amended by a total-derivative ‘boundary’ term. A way to determine whether such boundary term is required is to compute the ‘leakage’ of the symplectic current between two Cauchy slices [34]. Only for zero leakage is the symplectic product well-defined (i.e. independent of the Cauchy slice). In [34] this strategy is followed to obtain the boundary contribution from a horizon boundary. Here we follow the same strategy with spatial infinity as a boundary.

To proceed we need to specify fall-off conditions on the vector potential A_{μ} . We assume a power series expansion in ρ^{-1} with leading terms given by:

$$A_{\alpha} = O(\rho^0), \quad A_{\rho} = O(\rho^{-1}) \quad (4.2)$$

which is compatible with the fall-offs of $F_{\mu\nu}$ described in section 2.2. The leakage of the current (4.1) between two Cauchy slices $\tau = \tau_1$ and $\tau = \tau_2$ is then given by:

$$\lim_{\rho \rightarrow \infty} \int_{\tau_1}^{\tau_2} d\tau \int d^2 \hat{x} \omega_{\text{bulk}}^{\rho} = \int_{\Delta} d^3 V h^{\alpha\beta} \delta F_{\rho\alpha}^{-1} \wedge \delta A_{\beta}^0 \quad (4.3)$$

where $\Delta \subset \mathcal{H}^o$ is the region bounded by the spheres $\tau = \tau_1, \tau_2$ on \mathcal{H}^o . Written in terms of the vector potential, the leaking term reads:

$$h^{\alpha\beta} \delta F_{\rho\alpha}^{-1} \wedge \delta A_{\beta}^0 = -\delta D^{\alpha} A_{\rho}^{-1} \wedge \delta A_{\alpha}^0 \quad (4.4)$$

$$= -D^{\alpha} \left(\delta A_{\rho}^{-1} \wedge \delta A_{\alpha}^0 \right) + \delta A_{\rho}^{-1} \wedge \delta D^{\alpha} A_{\alpha}^0 \quad (4.5)$$

(indices in \mathcal{H}^o are raised with $h^{\alpha\beta}$). Since the first term in (4.5) is a total derivative, it can be cancelled by adding a boundary term to (4.1). The appropriate boundary term is

$$\omega_{\text{bdy}}^a := \partial_b \left(\sqrt{\eta} \delta A^a \wedge \delta A^b \right). \quad (4.6)$$

The second term in (4.5) however remains. This term can be eliminated by imposing appropriate gauge condition on the leading vector potential. For instance any gauge condition of the form

$$A_\rho^{-1} \propto D^\alpha A_\alpha^0 \quad (4.7)$$

will make the unwanted contribution to vanish. In fact, our interest is in studying large U(1) gauge parameter in Lorenz gauge. To leading order in $\rho \rightarrow \infty$ this gauge condition implies

$$2A_\rho^{-1} + D^\alpha A_\alpha^0 = 0, \quad (4.8)$$

which is a particular case of (4.7). To summarize: the symplectic structure associated to the $\tau = \text{const.}$ foliation, with fall-off conditions (4.2) and gauge condition (4.7) is given by:

$$\Omega = \int_{\Sigma_\tau} dS_a (\omega_{\text{bulk}}^a + \omega_{\text{bdy}}^a) \quad (4.9)$$

with ω_{bulk}^a and ω_{bdy}^a given in (4.1) and (4.6) respectively.⁴

4.2 Large U(1) gauge transformations and associated charges

We now consider large gauge transformations that are nontrivial at spatial infinity,

$$\Lambda(\rho, y) = \lambda(y) + O(\rho^{-1}). \quad (4.10)$$

The associated charges $Q[\lambda]$ are defined by the condition⁵

$$\delta Q[\lambda] = \Omega(\delta_\Lambda, \delta) \quad (4.11)$$

where δ_Λ is the action of the gauge transformation, eq. (2.4). The contribution to (4.11) due to the ‘bulk’ symplectic current becomes, upon using the field equations, the total derivative term

$$\omega_{\text{bulk}}^a(\delta_\Lambda, \delta) = -\delta \partial_b \left(\sqrt{\eta} \Lambda F^{ab} \right). \quad (4.12)$$

On the other hand, the contribution from ω_{bdy}^a is a total derivative to begin with:

$$\omega_{\text{bdy}}^a(\delta_\Lambda, \delta) = \delta \partial_b \sqrt{\eta} \left(\nabla^a \Lambda A^b - A^a \nabla^b \Lambda \right). \quad (4.13)$$

The integral over Σ_τ thus becomes a $\rho \rightarrow \infty$ surface integral. The fall-offs (4.2), (4.10) imply the charge is given by:

$$Q_\tau[\lambda] = \int_{C_\tau} dS_\alpha \sqrt{h} h^{\alpha\beta} \left(\partial_\beta \lambda A_\rho^{-1} - \lambda F_{\beta\rho}^{-1} \right), \quad (4.14)$$

⁴The fall-offs (4.2) imply a logarithmically divergent term in the ρ integral of (4.9). This would-be divergence however can be shown to be zero due to properties satisfied by the fields on \mathcal{H}^o . This and related points are discussed in subsection 4.3.

⁵The charge will depend on Λ only through its asymptotic value λ hence the notation $Q[\lambda]$.

where C_τ is the $\tau = \text{const.}$ surface on \mathcal{H}^o . For the fall-offs (4.2) we have $F_{\alpha\rho}^{-1} = \partial_\alpha A_\rho^{-1}$. Comparing with (2.23) we see that A_ρ^{-1} is the field ψ of the previous sections. Thus (4.14) can be written as:⁶

$$Q_\tau[\lambda] = \int_{C_\tau} dS_\alpha \sqrt{\hbar} h^{\alpha\beta} (\partial_\beta \lambda \psi - \lambda \partial_\beta \psi). \quad (4.15)$$

This has the form of a symplectic product of two scalar fields ψ and λ on \mathcal{H}^o . In section 2.2 we saw that both ψ and λ satisfy the wave equation on \mathcal{H}^o , (2.24) and (2.27) (the latter arising from the Lorenz gauge condition). It thus follows that (4.15) is independent of the slice C_τ . That is, the charges $Q_\tau[\lambda]$ are conserved. Finally, it is easy to verify that the $\tau \rightarrow \infty$ fall-offs (2.28), (3.9) imply (see appendix B.2)

$$\lim_{\tau \rightarrow \pm\infty} Q_\tau[\lambda] = Q_\pm[\lambda_\pm] \quad (4.16)$$

with $Q_\pm[\lambda_\pm]$ the charges defined in eqs. (3.3), (3.4) (just evaluate (4.14) for $\alpha = \tau$ and use $\sqrt{\hbar} h^{\tau\tau} = -(1 + \tau^2)^{3/2}$). The matching condition (3.2) can be seen as a consequence of the wave equation and fall-offs on λ , see appendix B.

4.3 Asymptotic properties of the symplectic structure

Let us denote by $\Omega_\tau^{\text{KG}}(\delta, \delta')$ the Klein-Gordon (KG) symplectic product on \mathcal{H}^o so that eq. (4.15) takes the form

$$Q_\tau[\lambda] = \Omega_\tau^{\text{KG}}(\lambda, \psi). \quad (4.17)$$

As discussed in appendix B, the fields λ and ψ have different $\tau \rightarrow \pm\infty$ fall-off behavior, associated to different ‘free data’ that can be prescribed in the asymptotic boundary of \mathcal{H}^o . This in turn implies that the KG symplectic product between λ ’s or ψ ’s fields vanish

$$\Omega_\tau^{\text{KG}}(\lambda, \lambda') = \Omega_\tau^{\text{KG}}(\psi, \psi') = 0. \quad (4.18)$$

In other words, the phase space of massless KG fields on \mathcal{H}^o has a “ (q, p) ” decomposition with λ -type fields playing the role of q and ψ -type fields the role of p .⁷ There are two interesting observations that follow.

The first one is in relation to the boundary phase space of gauge theories given in [38]. The authors consider gauge theories in finite spatial regions. In order to have good gluing properties of different regions, the authors are lead to introduce a boundary phase space consisting of a conjugate pair of normal electric field and gauge phase. Here we see the same type of boundary phase space: if λ is regarded as a dynamical variable, it follows that λ and ψ are canonically conjugated (with Q_τ expressing their symplectic product). In the language of [38], ψ plays the role of normal electric field and λ is the conjugated phase.

⁶An analogue charge is given in [35] for supertranslations in gravity. Our argument for charge conservation is the same as the one given there.

⁷From the point of view of the $\tau = 0$ slice, this decomposition is related to ‘parity conditions’ given in [36] (analogous to those in gravity [37]). See appendix B.2.C of [20] for a discussion in the context of massless scalars.

The second observation is regarding a potential logarithmic divergence in the symplectic product (4.9). Expressing the integral over Σ_τ in (4.9) as $\lim_{R \rightarrow \infty} \int_0^R d\rho \int d^2\hat{x}(\dots)$ one finds

$$\Omega_\tau(\delta, \delta') = \lim_{R \rightarrow \infty} \log(R) \Omega_\tau^{\text{KG}}(\delta\psi, \delta'\psi) + \text{finite}. \tag{4.19}$$

However, by virtue of (4.18) this would-be divergent piece vanishes.⁸

5 Magnetic charges

Dual ‘magnetic’ charges can be obtained by considering the previous charges with

$$F_{ab}^* := \frac{1}{2} \epsilon_{abcd} F^{cd} \tag{5.1}$$

playing the role of F_{ab} . From the null infinity perspective we have

$$\overset{-2}{F}^*_{ru} = \frac{1}{2} \epsilon^{AB} \overset{0}{F}_{AB} \tag{5.2}$$

and similarly for \mathcal{I}^- . The magnetic versions of the conservation law (3.1) and charges (3.3), (3.4) are defined by the replacement $F_{ab} \rightarrow F_{ab}^*$. For instance, we have

$$\tilde{Q}_+[\lambda] = \int_{\mathcal{I}^+} dud^2V \lambda(\hat{x}) \partial_u \overset{-2}{F}^*_{ru}(u, \hat{x}) - \int_{S^2} d^2V \lambda(\hat{x}) \overset{-2}{F}^*_{ru}(u = \infty, \hat{x}) \tag{5.3}$$

as the magnetic analogue of (3.6). Using (5.2) we see that the first term in (5.3) is the magnetic ‘soft’ charge that features in Weinberg’s soft theorem [30]. In appendix C we show that the field equations at time-like infinity imply that the second term in (5.3) vanishes and there is no ‘hard’ contribution to the magnetic charge.

At spatial infinity, the starred field strength is given by

$$\overset{0}{F}^*_{\alpha\beta} = \epsilon_{\alpha\beta}{}^\gamma{}^\delta \overset{-1}{F}_{\gamma\delta} \tag{5.4}$$

$$\overset{-1}{F}^*_{\alpha\rho} = \frac{1}{2} \epsilon_\alpha{}^{\beta\gamma} \overset{0}{F}_{\beta\gamma}. \tag{5.5}$$

The definition of $\tilde{\psi}$ in (2.25) then implies

$$\overset{-1}{F}^*_{\alpha\rho} = D_\alpha \tilde{\psi}. \tag{5.6}$$

Thus, the replacement $F_{\mu\nu} \rightarrow F_{\mu\nu}^*$ corresponds to $\psi \rightarrow \tilde{\psi}$ (and $\tilde{\psi} \rightarrow -\psi$). In particular, the conserved magnetic charges at spatial infinity are

$$\tilde{Q}_\tau[\lambda] = \int_{C_\tau} dS_\alpha \sqrt{h} h^{\alpha\beta} \left(\partial_\alpha \lambda \tilde{\psi} - \lambda \partial_\alpha \tilde{\psi} \right). \tag{5.7}$$

As for the electric charges, they satisfy $\lim_{\tau \rightarrow \pm\infty} \tilde{Q}_\tau[\lambda] = \tilde{Q}_\pm[\lambda_\pm]$.

⁸To simplify the discussion we have omitted a second logarithmic divergence proportional to $\sim \int dS_\alpha \sqrt{h} h^{\alpha\beta} \delta \overset{0}{F}_{\alpha\beta} \wedge \delta \overset{0}{A}_\beta$. This can also be shown to be zero by entirely similar arguments.

6 Summary and outlook

The first goal of this paper was to provide a dynamical explanation for the “conservation laws”

$$\overset{-2}{F}_{ru}(u = -\infty, \hat{x}) = \overset{-2}{F}_{rv}(v = \infty, -\hat{x}) \tag{6.1}$$

recently proposed by Strominger and collaborators in relation to Weinberg’s soft photon theorem. Our treatment was purely classical; we did not need worry about possible subtleties in the definition of the quantum counterpart of (6.1) that is actually used in connection with the soft photon theorem.

It has long been known [21, 22] that an appropriate way to think of spatial infinity is as a unit time-like hyperboloid \mathcal{H}^o parametrizing all asymptotic directions of spatial geodesics. Information at the future end of \mathcal{I}^- is propagated along \mathcal{H}^o and reaches the past end of \mathcal{I}^+ . In this way, condition (6.1) appears as a consequence of the field equations at \mathcal{H}^o . We now summarize the main ingredients that went into our proof of eq. (6.1). We first assumed standard $1/r$ and $1/\rho$ power expansions of the field strength at null and spatial infinity respectively. We then argued that in the absence of massless charges, the asymptotic field at null and spatial infinity should satisfy the free-field equations. The free-field equations at \mathcal{I}^+ fix the $u \rightarrow -\infty$ fall-off of the coefficients in the $1/r$ expansion (once the ‘free data’ fall-off is specified). The free-field equations at \mathcal{H}^o fix the $\tau \rightarrow \infty$ fall-off of the coefficients in the $1/\rho$ expansion up to an ambiguity that can be resolved by demanding consistency with the null-infinity expansion. This consistency allows one to write the l.h.s. of (6.1) as a $\tau \rightarrow \infty$ limit of a field ψ on \mathcal{H}^o . Similarly, the r.h.s. of (6.1) can be written as the $\tau \rightarrow -\infty$ limit of ψ . The dynamical equation satisfied by ψ on \mathcal{H}^o implies then eq. (6.1). We note that up to this point, the discussion is phrased purely in terms of the field strength. In particular, it does not refer to any specific gauge fixing condition.

Our second goal was to explore large U(1) gauge symmetries and charges from the perspective of \mathcal{H}^o . At future/past infinities these charges are known to be given by a smeared version of (1.1). In order to get ‘finite time’ versions of these charges, we considered a spacetime foliation of hypersurfaces Σ_τ that intersect \mathcal{H}^o on constant τ spheres. After imposing an asymptotic Lorenz gauge condition, we obtained the canonical charge $Q_\tau[\lambda]$ associated to a large gauge symmetry λ . The properties of the field equations at spatial infinity imply that $Q_\tau[\lambda]$ is conserved and its $\tau \rightarrow \pm\infty$ limit given by the known future/past infinity charges. While deriving these charges we encountered several interesting properties of the symplectic structure that may be of interest in their own: the need to include a boundary term, the relation with the boundary phase space of [38], and the absence of logarithmic divergences as a consequence of the properties of the fields on \mathcal{H}^o . We emphasize that in our second goal we worked with the vector potential in an asymptotic Lorenz gauge condition.

Many future directions appear in sight. Extensions of the ideas presented here to (at least perturbative) gravity seem within reach.⁹ Indeed many of the ingredients can already be found in the literature: the supertranslation charges at spatially infinity (in an

⁹The case of supertranslation charges in gravity was recently discussed in [39].

asymptotic harmonic-like gauge) given in [35] are completely analogue to the charges $Q_\tau[\lambda]$ found here. The extensions of supertranslations and superrotations from null to time-like infinity described in [40] have a natural counterpart in the analogue problem of going from null to space-infinity. Finally, it would be interesting to explore possible connections to studies in Minkowski holography [41–44].

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A More on fall-offs and asymptotic field equations

A.1 Eq. (2.13)

A convenient gauge to solve the free Maxwell equations is radial gauge $A_r = 0$ (see for instance [31]). In this gauge, the equation $\nabla_b F^{ub} = 0$ implies

$$\partial_r (r^2 F_{ru}) = \partial_r D^A A_A, \tag{A.1}$$

which gives F_{ru} in terms of A_A up to an integration ‘constant’ $k(u, \hat{x})$. We assume a r^{-n} power expansion for A_A and A_u compatible with (2.9), (2.8). Eq. (2.10) then implies that the aforementioned integration ‘constant’ is u -independent. We thus have

$$F_{ru} = r^{-2} (D^A A_A + k(\hat{x})). \tag{A.2}$$

Next, we look at the equation $\nabla_b F^{Ab} = 0$. Upon expanding in $1/r$ and using (A.2) it implies:

$$-2(n+1)\partial_u \bar{A}_A^{-(n+1)} + (-D^2 + n^2 + n + 1) \bar{A}_A^{-n} - \delta_{n0} \partial_A k = 0, \quad n = 0, 1, \dots \tag{A.3}$$

This can be integrated to give \bar{A}_A^{-n} in terms of $\bar{A}_A^{-(n-1)}$ (and $k(\hat{x})$). If $\bar{A}_A^0 = O(|u|^0)$ as in eq. (2.12) this implies

$$\bar{A}_A^{-n} = O(|u|^n). \tag{A.4}$$

Using (A.4) in (A.2) we obtain eq. (2.13).

A.2 Eq. (2.20)

Free field Maxwell equations at spatial infinity take simple form when written in terms of the vector potential in Lorenz gauge. For \bar{A}^{-n} one finds the equation

$$D^\alpha D_\alpha \bar{A}_\rho^{-n} + (n-1)(n-3) \bar{A}_\rho^{-n} = 0 \tag{A.5}$$

this imply that \bar{A}_ρ^{-n} can be $O(\tau^{1-n})$ or $O(\tau^{n-3})$ at $\tau \rightarrow \pm\infty$, which corresponds to $O(\tau^{-n})$ or $O(\tau^{n-4})$ behavior of $\bar{F}_{\rho\tau}^{-n}$. For $n \geq 2$ the leading behavior is $O(\tau^{n-4})$ which corresponds to eq. (2.20). However for $n = 1$ the leading piece is the $O(\tau^{-n})$ one. One could say that the demand of this leading piece to vanish corresponds to the condition of having well-defined charges, i.e. only then eqs. (3.7), (3.8) make sense. We now provide an argument as to how this condition (absence of $O(\tau^{-1})$ term in the $\tau \rightarrow \infty$ asymptotic of $\bar{F}_{\rho\tau}^{-1}$) arises from the assumed behavior at null infinity.

In the previous subsection we saw that $\bar{F}_{ru}^{-k} = O(|u|^{k-2})$ at $u \rightarrow \infty$. Let us assume a $1/u$ power expansion at $u \rightarrow \infty$ for the subleading terms so that

$$F_{ru}(r, u, \hat{x}) = \sum_{k=2}^{\infty} \frac{u^{k-2}}{r^k} \sum_{l=0}^{\infty} \frac{1}{u^l} \left(\bar{F}_{ru}^{-k} \right)^{(l)}(\hat{x}). \tag{A.6}$$

Consider now the above expression in the (ρ, τ) coordinates adapted to spatial infinity. Substituting

$$r = \rho\sqrt{1 + \tau^2} = \rho\tau (1 + O(\tau^{-2})), \tag{A.7}$$

$$u = \rho(\tau - \sqrt{1 + \tau^2}) = -\frac{\rho}{2\tau} (1 + O(\tau^{-2})) \tag{A.8}$$

in (A.6) and using the change of coordinates relation

$$F_{\rho\tau} = \frac{\rho^2}{r} F_{ru} = \frac{\rho}{\tau} (1 + O(\tau^{-2})) F_{ru} \tag{A.9}$$

one finds (after reorganizing the sums) that indeed the leading $1/\tau$ power of each $1/\rho$ term corresponds to $\bar{F}_{\rho\tau}^{-n} = O(\tau^{n-4})$.

A.3 Eqs. (3.7), (3.8)

The equality we wish to study deals with a priori different limits. The r.h.s. of eq. (3.7) corresponds to:

$$\lim_{\tau \rightarrow \infty} \tau^3 \left[\lim_{\rho \rightarrow \infty} \rho F_{\rho\tau}(\rho, \tau, \hat{x}) \right] =: \left(\bar{F}_{\rho\tau}^{-1} \right)^{(-3)}(\hat{x}) \tag{A.10}$$

whereas the l.h.s. is:

$$\lim_{u \rightarrow -\infty} \left[\lim_{r \rightarrow \infty} r^2 F_{ru}(r, u, \hat{x}) \right] =: \left(\bar{F}_{ru}^{-2} \right)^{(0)}(\hat{x}) \tag{A.11}$$

The idea is to capture the above, a priori different, limits as a single limit along an appropriately chosen curve $\gamma(s)$ in spacetime,

$$s \mapsto \gamma(s) = (t(s), r(s), \hat{x}). \tag{A.12}$$

A curve that will do the jobs is:

$$\rho(s) = s^2, \quad \tau(s) = s, \tag{A.13}$$

whose form in (r, u) coordinates can be obtained from (A.7), (A.8):

$$r(s) = s^3 + O(s), \quad u(s) = -\frac{s}{2} + O(1/s). \quad (\text{A.14})$$

The intuitive picture of this curve is as follows. Eq. (A.10) tell us to first take $\rho \rightarrow \infty$ before taking $\tau \rightarrow \infty$. The curve (A.13) accomplishes this goal by making $\rho(s)$ diverge faster than $\tau(s)$ as $s \rightarrow \infty$. Similarly eq. (A.11) tell us that ‘ r should go to infinity faster than u goes to (minus) infinity’. This is precisely the $s \rightarrow \infty$ behavior of the curve in (r, u) coordinates (A.14).

The fall-offs described in section 2 ensure this intuitive picture is correct and that one can replace the limits in eqs. (A.10), (A.11) by a single $s \rightarrow \infty$ limit. For (A.10), it is easy to verify that the expansion (2.19) together with τ fall-off (2.20) imply

$$\lim_{s \rightarrow \infty} s^5 F_{\rho\tau}(x = \gamma(s)) = \left(F_{\rho\tau}^{-1} \right)^{(-3)}(\hat{x}). \quad (\text{A.15})$$

Similarly, it is easy to verify that the $r \rightarrow \infty$ expansion (2.9) together with the u fall-offs (2.13) imply

$$\lim_{s \rightarrow \infty} s^6 F_{ru}(x = \gamma(s)) = \left(F_{ru}^{-2} \right)^{(0)}(\hat{x}). \quad (\text{A.16})$$

Finally, the relation between $F_{\rho\tau}$ and F_{ru} from the change of coordinates, $F_{\rho\tau} = \frac{\rho^2}{r} F_{ru}$, implies

$$F_{\rho\tau} = (s + O(1/s)) F_{ru} \quad (\text{A.17})$$

along the $\gamma(s)$ curve. Substituting (A.17) in (A.15) we see that this limit is precisely the limit (A.16). It then follows that $\left(F_{\rho\tau}^{-1} \right)^{(-3)}(\hat{x}) = \left(F_{ru}^{-2} \right)^{(0)}(\hat{x})$. This establishes eq. (3.7). Eq. (3.8) can be shown by an analogous argument.

B Wave equation on \mathcal{H}^o

B.1 Green’s functions

In this section we describe the solutions to the wave equation

$$D^\alpha D_\alpha \psi = 0 \quad (\text{B.1})$$

in terms of asymptotic data at one asymptotic boundary of \mathcal{H}^o . For definitiveness we consider data at the past asymptotic boundary, $\tau \rightarrow -\infty$ but similar considerations apply to future asymptotic boundary data. By analyzing the wave equation (B.1) in the $\tau \rightarrow -\infty$ limit, one concludes there are two types of asymptotic solutions:

$$\lambda(\tau, \hat{x}) = \lambda_-(\hat{x}) + O(\tau^{-2} \ln |\tau|) \quad (\text{B.2})$$

$$\psi(\tau, \hat{x}) = \tau^{-2} \psi_-(\hat{x}) + O(\tau^{-4}) \quad (\text{B.3})$$

The leading terms $\lambda_-(\hat{x})$, $\psi_-(\hat{x})$ are unconstrained and represent free data in terms of which subleading terms are determined. As indicated in the notation, the first type of

solution is relevant for large gauge parameters whereas the second type is relevant for the leading component of the field strength.¹⁰ We wish to obtain integral expressions for the solutions $\lambda(y)$, $\psi(y)$ in terms of the corresponding asymptotic data:

$$\lambda(y) = \int d^2V' G^{(0)}(y; \hat{x}') \lambda_-(\hat{x}') \quad (\text{B.4})$$

$$\psi(y) = \int d^2V' G^{(2)}(y; \hat{x}') \psi_-(\hat{x}'), \quad (\text{B.5})$$

where the G 's are appropriate Green's functions. In the following we describe these Green's functions. For that purpose let us introduce some notation. Given $y \in \mathcal{H}^o$ and $\hat{x}' \in S^2$ define:

$$Y^\mu := \left(\tau, \sqrt{1 + \tau^2} \hat{x} \right) \quad (\text{B.6})$$

$$\sigma := \tau + \sqrt{1 + \tau^2} \hat{x} \cdot \hat{x}', \quad (\text{B.7})$$

that is, $y \in \mathcal{H}^o$ defines a point Y^μ in the unit hyperboloid of Minkowski space. \hat{x}' defines the point on the past light cone: $(-1, \hat{x}')^\mu$. σ is then the inner product between these two vectors.

We now study each Green's function separately.

B.1.1 Green's function for eq. (B.2)

The Green's function in (B.4) is a Lorentzian version of the Green's function used in [26] to obtain the gauge parameter at time-like infinity.¹¹ Taking the appropriate analytic continuation of the expression given in [26] yields:

$$G^{(0)}(y; \hat{x}') = -\frac{1}{4\pi\sigma^2}. \quad (\text{B.8})$$

Let us verify that indeed (B.8) satisfies the desired requirements. The action of the wave operator $D^\alpha D_\alpha$ on a function $f(\sigma)$ can be easily calculated by expressing $D^\alpha D_\alpha$ in terms of the Minkowski space d'Alembertian (see [46] for similar treatment at time-like infinity). One obtains:

$$D^\alpha D_\alpha f(\sigma) = -\sigma^2 f''(\sigma) - 3\sigma f'(\sigma). \quad (\text{B.9})$$

It then follows that $f(\sigma) = \sigma^{-2}$ and hence $G^{(0)}(y; \hat{x}')$ satisfies the wave equation. We now verify the boundary condition. The $\tau \rightarrow -\infty$ behavior of σ is

$$\sigma \xrightarrow{\tau \rightarrow -\infty} \tau(1 - \hat{x} \cdot \hat{x}') - \frac{\hat{x} \cdot \hat{x}'}{2\tau} + \dots \quad (\text{B.10})$$

From which we have

$$G^{(0)}(y; \hat{x}') \xrightarrow{\tau \rightarrow -\infty} \begin{cases} O(\tau^{-2}) & \text{if } \hat{x} \neq \hat{x}' \\ O(\tau^2) & \text{if } \hat{x} = \hat{x}' \end{cases} \quad (\text{B.11})$$

¹⁰Recall ψ in eq. (2.23) is defined up to an overall constant. For the following argument we choose such constant so that $k_- = 0$ in eq. (3.9).

¹¹The Green's functions of both signatures are particular cases of the more general discussion given in [41].

On the other hand one can verify that $\int d^2V'G^{(0)}(y;\hat{x}') = 1$. It then follows that $G^{(0)}(y;\hat{x}') \xrightarrow{\tau \rightarrow -\infty} \delta^{(2)}(\hat{x}, \hat{x}')$ which then imposes the boundary condition (B.2). The solution given by (B.4), (B.8) allow us to explore the $\tau \rightarrow +\infty$ limiting value. In this case

$$\sigma \xrightarrow{\tau \rightarrow +\infty} \tau(1 + \hat{x} \cdot \hat{x}') + \frac{\hat{x} \cdot \hat{x}'}{2\tau} + \dots \quad (\text{B.12})$$

which now implies $G^{(0)}(y;\hat{x}') \xrightarrow{\tau \rightarrow -\infty} \delta^{(2)}(\hat{x}, -\hat{x}')$. We thus recover the antipodal identification (3.2) of the gauge parameter at plus and minus infinity.

B.1.2 Green's function for eq. (B.3)

To express ψ in terms of given asymptotic data, we consider the Kirchhoff integral representation (see for instance section 4.3 of [47]):

$$\psi(y) = - \int_C dS'_\alpha \sqrt{h} h^{\alpha\beta}(y') (G_R(y, y') \partial'_\beta \psi(y') - \partial'_\beta G_R(y, y') \psi(y')) \quad (\text{B.13})$$

where C is a Cauchy slice of \mathcal{H}^o and $G_R(y, y')$ the retarded Green's function. The idea is to consider (B.13) with C a $\tau' = \text{const.}$ surface and take the limit $\tau' \rightarrow -\infty$. In such case eq. (B.13) takes the form:

$$\psi(y) = \lim_{\tau' \rightarrow -\infty} |\tau'|^3 \int_{\tau' = \text{const.}} d^2V' (G_R(y, y') \partial_{\tau'} \psi(y') - \partial_{\tau'} G_R(y, y') \psi(y')). \quad (\text{B.14})$$

To proceed we need the expression for $G_R(y, y')$. For y inside the future cone of y' , it has to satisfy the wave equation (B.1) and be symmetric with respect to the isotropy group of y' . The solution is easily found in terms of

$$P = Y^\mu Y'_\mu \quad (\text{B.15})$$

(see e.g. [45]) with Y^μ and Y'^μ defined in (B.6) for y and y' respectively. One finds:

$$G_R(y, y') = \frac{1}{2\pi} \theta(\tau - \tau') \theta(P - 1) \frac{P}{\sqrt{P^2 - 1}} \quad (\text{B.16})$$

where θ is the step function that imposes y to be inside the cone of y' and $\tau > \tau'$. It is easy to verify that for $y \rightarrow y'$ (B.16) approaches the flat space retarded Green's function. It now remains to study the $\tau' \rightarrow -\infty$ behaviour of $G_R(y, y')$ in order to evaluate (B.14). When $\tau' \rightarrow -\infty$ one has

$$P = -\tau' \sigma + O(\tau'^{-1}), \quad (\text{B.17})$$

and

$$G_R(y, y') = \theta(\sigma) + O(\tau^{-1}) \quad (\text{B.18})$$

with σ as defined in (B.7). Using (B.3) and (B.18) in eq. (B.14) one obtains:

$$\psi(y) = \frac{1}{\pi} \int d^2V' \theta(\sigma) \psi_-(\hat{x}'). \quad (\text{B.19})$$

It is instructive to verify that (B.19) is indeed a solution to (B.1) with asymptotic condition (B.2). First notice that, as in the λ case, the dependance on y is through σ . We can thus use (B.9) to evaluate the wave operator on (C.5):

$$D^\alpha D_\alpha \psi(y) = -\frac{1}{\pi} \int d^2 V' (\sigma^2 \delta'(\sigma) + 3\sigma \delta(\sigma)) \psi_-(\hat{x}') \tag{B.20}$$

with $\delta = \theta'$ the delta function. For regular data $\psi_-(\hat{x}')$ each term in (B.20) vanishes.

We now study the asymptotic behavior of $\psi(y)$. Eq. (C.5) tells one is integrating $\psi_-(\hat{x}')$ over the portion of the \hat{x}' sphere given by

$$\hat{x} \cdot \hat{x}' > -\frac{\tau}{\sqrt{1 + \tau^2}}. \tag{B.21}$$

When $\tau \rightarrow -\infty$, the region (B.21) becomes a disk of radius $|\tau|^{-1}$ centered at $\hat{x}' = \hat{x}$. Thus (C.5) becomes $\psi(y) \approx \frac{1}{\pi} (\pi \tau^{-2} \psi_-(\hat{x}))$ and we recover the boundary condition (B.3).

At the other extreme, when $\tau \rightarrow \infty$ the integration region becomes the whole sphere minus a disk of radius τ^{-1} centered at $\hat{x}' = -\hat{x}$. Thus, in this limit the solution becomes

$$\psi(y) \xrightarrow{\tau \rightarrow +\infty} \frac{1}{\pi} \left(\int_{S^2} d^2 V' \psi_-(\hat{x}') - \pi \tau^{-2} \psi_-(-\hat{x}) \right), \tag{B.22}$$

which is of the form (3.9) with a nonzero k_+ and $\psi_+(\hat{x}) = -\psi_-(-\hat{x})$. This establishes the relation (3.12).

B.2 Eq. (4.16)

Let us for definitiveness consider the case $\tau \rightarrow +\infty$. The general $\tau \rightarrow \infty$ fall-offs for λ and ψ are

$$\lambda(\tau, \hat{x}) = \lambda_+(\hat{x}) + \frac{\ln \tau}{\tau} \mu_+(\hat{x}) + O(\tau^{-3}) \tag{B.23}$$

$$\psi(\tau, \hat{x}) = k_+ + \frac{1}{\tau^2} \psi_+(\hat{x}) + O(\tau^{-4}). \tag{B.24}$$

From the wave equation $D^\alpha D_\alpha \lambda = 0$ one finds

$$\mu_+ = -\frac{1}{2} \Delta \lambda_+. \tag{B.25}$$

In the $\tau \rightarrow \infty$ limit, the integrand of (4.15) is then given by

$$\sqrt{\hbar} h^{\tau\tau} (\partial_\tau \lambda \psi - \lambda \partial_\tau \psi) = (\tau \ln \tau - \tau) k_+ \mu_+ - 2\lambda_+ \psi_+ + O(\tau^{-1}). \tag{B.26}$$

The term proportional to $k_+ \mu_+$ integrates to zero by virtue of eq. (B.25) and one finds

$$Q_{\tau=\infty} = -2 \int_{S^2} d^2 V \lambda_+ \psi_+, \tag{B.27}$$

which coincides with (3.3) due to eq. (3.10).

C Laplace and Poisson equations on \mathcal{H}^+

C.1 Solutions to leading order equations

In this section we study solutions to Laplace's and Poisson's equations on \mathcal{H}^o :

$$D^\alpha D_\alpha \tilde{\psi} = 0, \quad (\text{C.1})$$

$$-D^\alpha D_\alpha \psi = \frac{-3}{j_\tau}. \quad (\text{C.2})$$

By similar arguments to those given in section A.2 one can show that the $\rho \rightarrow \infty$ behavior of $\psi, \tilde{\psi}$ is (up to arbitrary additive constants that we set to zero)

$$\psi, \tilde{\psi} = O(\rho^{-2}). \quad (\text{C.3})$$

On the other hand, Laplace's equation (C.1) on \mathcal{H}^o imply a $O(\rho^0)$ or $O(\rho^{-2})$ behavior at $\rho \rightarrow \infty$. The second kind of behavior however yields solutions that are singular at $\rho = 0$ (this can for instance be established by looking at the solutions with given spherical harmonic angular dependence). Thus eqs. (C.1) and (C.3) imply

$$\tilde{\psi} = 0. \quad (\text{C.4})$$

We not turn to Poisson's equation (C.2). The solution for ψ can be written as

$$\psi(y) = \int d^3V' \mathcal{G}(y; y') \frac{-3}{j_\tau}(y'), \quad (\text{C.5})$$

with $\mathcal{G}(y; y')$ the appropriate Green's function. This Green's function can be found by similar methods as those leading to eq. (B.16) resulting in,

$$\mathcal{G}(y; y') = \frac{1}{4\pi} \left(\frac{Y \cdot Y'}{\sqrt{(Y \cdot Y')^2 - 1}} - 1 \right) \quad (\text{C.6})$$

where Y^μ and Y'^μ are Minkowski vectors associated to $y, y' \in \mathcal{H}^+$, i.e.

$$Y^\mu = \left(\sqrt{1 + \rho^2}, \rho \hat{x} \right) \quad (\text{C.7})$$

and $Y \cdot Y'$ their inner product with $(+, -, -, -)$ sign convention. One can verify that (C.6) approaches the standard flat-space Green's function in the limit $y \rightarrow y'$. The “ -1 ” in (C.6) ensures the possible additive constant on ψ is set to zero.

The solution to Laplace's equation for λ (2.39) with given boundary value $\lambda_+(\hat{x})$ was discussed in [26]. It is given by

$$\lambda(y) = \int d^2V' G(y; \hat{x}') \lambda_+(\hat{x}'), \quad (\text{C.8})$$

where the Green's function is

$$G(y; \hat{x}') = \frac{1}{4\pi \sigma(y, \hat{x}')^2} \quad (\text{C.9})$$

with

$$\sigma(y, \hat{x}') = \sqrt{1 + \rho^2} - \rho \hat{x} \cdot \hat{x}' \quad (\text{C.10})$$

the inner product between Y^μ and $(1, \hat{x}')^\mu$.

C.2 Time infinity contribution to charges

In this section we evaluate the boundary terms in eqs. (3.6), (5.3):

$$Q^{i+}[\lambda_+] := - \int_{S^2} d^2V \lambda_+(\hat{x}) \bar{F}_{ru}^{-2}(u = \infty, \hat{x}), \quad (\text{C.11})$$

$$\tilde{Q}^{i+}[\lambda_+] := -\frac{1}{2} \int_{S^2} d^2V \lambda_+(\hat{x}) \epsilon^{AB} \bar{F}_{AB}^0(u = \infty, \hat{x}). \quad (\text{C.12})$$

By a similar argument to the one given in section A.3, one finds:

$$\bar{F}_{ru}^{-2}(u = \infty, \hat{x}) = \lim_{\rho \rightarrow \infty} \rho^3 \bar{F}_{\rho\tau}^{-1}(\rho, \hat{x}), \quad (\text{C.13})$$

$$\frac{1}{2} \epsilon^{AB} \bar{F}_{AB}^0(u = \infty, \hat{x}) = \lim_{\rho \rightarrow \infty} \rho^3 \bar{F}_{\rho\tau}^{*-1}(\rho, \hat{x}). \quad (\text{C.14})$$

Recalling the relation with $\psi, \tilde{\psi}$, and using (C.3) this allow us to write the time-infinity charges as

$$Q^{i+}[\lambda_+] = \lim_{\rho \rightarrow \infty} 2\rho^2 \int_{S^2} d^2V \lambda_+(\hat{x}) \psi(\rho, \hat{x}), \quad (\text{C.15})$$

$$\tilde{Q}^{i+}[\lambda_+] = \lim_{\rho \rightarrow \infty} 2\rho^2 \int_{S^2} d^2V \lambda_+(\hat{x}) \tilde{\psi}(\rho, \hat{x}). \quad (\text{C.16})$$

From (C.4) we immediately conclude that $\tilde{Q}^{i+}[\lambda_+] = 0$. We now show that $Q^{i+}[\lambda_+]$ coincides with the time-infinity charge obtained in [26] by covariant phase space methods. The expression given in [26] is

$$Q_{\mathcal{H}^+}[\lambda] = \int_{\mathcal{H}^+} d^3V \lambda(y) \bar{j}_\tau^{-3}(y), \quad (\text{C.17})$$

with λ the gauge parameter at \mathcal{H}^+ given in eq. (C.8). To see that (C.15) and (C.17) coincide, we consider the $\rho \rightarrow \infty$ limit of $\psi(y)$ as given in (C.5). In this limit

$$Y \cdot Y' = \rho \sigma(y', \hat{x}) + O(\rho^{-1}), \quad (\text{C.18})$$

$$\frac{Y \cdot Y'}{\sqrt{(Y \cdot Y')^2 - 1}} - 1 = \frac{1}{2\rho^2 \sigma(y', \hat{x})^2} + O(\rho^{-4}), \quad (\text{C.19})$$

and so

$$\mathcal{G}(y; y') = \frac{1}{2\rho^2} G(y'; \hat{x}) + O(\rho^{-4}) \quad (\text{C.20})$$

where \mathcal{G} and G are the Green's function given in eqs. (C.6) and (C.9) respectively. Using the expansion (C.20) in (C.15), (C.5) one finds:

$$Q^{i+}[\lambda_+] = \int d^2V \lambda_+(\hat{x}) \int d^3V' G(y'; \hat{x}) \bar{j}_\tau^{-3}(y') \quad (\text{C.21})$$

$$= \int d^3V' \lambda(y') \bar{j}_\tau^{-3}(y') \quad (\text{C.22})$$

where in the second line we used the expression (C.8) for the gauge parameter on \mathcal{H}^+ . We thus conclude that $Q^{i+}[\lambda_+]$ coincides with $Q_{\mathcal{H}^+}[\lambda]$ given in [26].

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