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On evolutionarily stable strategies and replicator dynamics in asymmetric two-population games

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Abstract

We analyze the main dynamical properties of the evolutionarily stable strategy ESS for asymmetric two-population games of finite size in its corresponding replicator dynamics. We introduce a defnition of ESS for two-population asymmetric games and a method of symmetrizing such an asymmetric game. Then, we show that every strategy profile of the asymmetric game corresponds to a strategy in the symmetric game, and that every Nash equilibrium (NE) of the asymmetric game corresponds to a (symmetric) NE of the symmetric version game. So, we study (standard) replicator dynamics for the asymmetric game and define corresponding (non-standard) dynamics of the symmetric game.

JEL classification: C72, C73, C79
Keywords: Asymmetric game; Evolutionary games; ESS; Replicator dynamics.

Resumen

Presentamos una extensión del concepto de estrategias evolutivamente estables al caso de juegos asimétricos. El objetivo de esta extensión es el de aprovechar las propiedades bien conocidas de estas estrategias en el caso simétrico y su relación con los equilibrios de la dinámica del replicador en este tipo de juegos, para analizar las propiedades dinámicas de dichas estrategias, cuando las ecuaciones diferenciales que rigen la evolución de las poblaciones no surgen de juegos simétricos. Para esto se crea una versión simétrica para cada juego asimétrico, y se comprueba que las estrategias evolutivamente estables de los juegos asimétricos siguen siendo evolutivamente estables para la versión simétrica y que se conservan algunas de las propiedades de estabilidad cuando se vuelve al caso asimétrico.

Clasificación JEL : C72, C73, C79
Palabras claves: estrategias evolutivamente estables, dinámica del replicador

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1 Introduction

Evolutionary dynamics was originally motivated by biology, then for economics and concerns pairwise random matching of individuals drawn from a single infinitely large population, and usually playing a symmetric game. Evolutionary stability, introduced by Maynard Smith and Price (1973), is a criterion for the robustness of an incumbent strategy against the entry of individuals or mutants using a different strategy. The framework considered is a conflict within a homogenous population (a symmetric game). A game in normal form is symmetric if all players have the same strategy set, and the payoff to playing a given strategy depends only on the strategies being played, not on who plays them.

Nevertheless, many economic applications come from attention for multi-population rather than single-population dynamics on asymmetric environments. So, in most applications, the game is not going to be symmetric and involve at least two players with different strategies and each player role is represented by a different population in the spirit of Nash's (1950) "mass action interpretation" where each type of player being drawn from his or her "player-role population". For instance, the player roles may be those individuals of buyers and sellers, incumbents and entrants in oligopolistic markets, workers and firms, or the social relationships between migrants and residents; all of them with non-homogeneous behaviors about the state of the economy or different attitudes towards - and perceptions about - development efforts or environmental quality of the state of the economy and so forth.

Recall that, from the framework of symmetric games, there is a seminal refinement of the Nash equilibrium ($\mathcal{NE}$) concept that is the notion of Evolutionarily Stable Strategy ($\mathcal{ESS}$) (see Maynard Smith and Price (1973), Maynard Smith (1974)). From it, we known that every $\mathcal{ESS}$ is at the same time, a stable strategy against mutants, i.e., is robust when is invaded by a small population playing a different strategy, and asymptotically stable steady state in the associated replicator dynamics. Hence, the relationship between $\mathcal{NE}$, $\mathcal{ESS}$ and the steady states ($\mathcal{SS}$) on this replicator dynamic, are well known (see Weibull (1995)).

Hence, in this paper, we consider the evolution of two populations facing a conflictive situation being modeled by an asymmetric normal form game. Analyzing the evolution and stability of the behaviors of the populations involved in asymmetric games is the main purpose of this work. In this vein, we should symmetrize the asymmetric game because it give us, the possibility to characterize the $\mathcal{ESS}$ using the well known properties of these strategies for the cases of symmetric games.

Then, we extend the concept of $\mathcal{ESS}$ for asymmetric two-population games, equivalently in the definition of Selten (1980) and Samuelson (1998) but in those papers was not analyzed the evolutionary dynamics of such a population. More close to our argument is the work of Fishman (2008), nevertheless our approach is quite different, since by symmetrizing the game we get the advantage of generalizing the standard definition of $\mathcal{ESS}$ and its relationship existing between stability of the dynamical equilibria corresponding to the replicator dynamics, and their strategic stability for the case of asymmetric games. Note that, much of the topic of this paper can be generalized for cases of finite asymmetric populations on $n > 2$, however to simplify notation, generally, we shall consider the case of two-player asymmetric populations.

To sum up, our approach allow us to characterize in an unified way the main characteristics of the $\mathcal{ESS}$ for the asymmetric cases. Following this approach it is straightforward to see that in asymmetric games, a strategic profile is an $\mathcal{ESS}$ if and only if is a strict Nash equilibrium (see Balkenborg and Schlag (1995), (2007); Samuelson (1998); Selten (1980); Weibull (1995)) and that every $\mathcal{ESS}$ is an asymptotically stable steady state of the replicator dynamic (see Retchkiman (2007); Samuelson and Zhang (1992)) and other results.

The paper is organized as follows. Section 2 draws the notation and basic definitions to set up the baseline model, namely a two-player asymmetric normal-form game. Section 3 defines the $\mathcal{ESS}$ for our model. In section 4, we introduce the symmetric version of an asymmetric two population game. Section 5 studies the dynamics from our model. Section 6 states the relationships between $\mathcal{ESS}$, $\mathcal{NE}$ and $\mathcal{SS}$. Section 7 draws some concluding remarks.

2 The model

Let us denote by G a normal-form (strategic) game with a player set composed by individuals that comprise τ populations, namely residents, R, and migrants, M: $\tau = \{R, M\}$. Each population splits in different clubs denoted by n_i^τ and $i = 1, ..., k_\tau$, i.e. $(n_1^R, ..., n_{k_R}^R)$ and $(n_1^M, ..., n_{k_M}^M)$. The split depends on the strategy agents play or the behavior that agents follow. Strategies are in correspondence with the clubs, individuals belonging to the n_i^τ club will be called n_i-strategists. Thus, the set S^τ of pure strategies are: $S^R = \left\{n_1^R, ..., n_k^R\right\}$ and $S^M = \left\{n_1^M, ..., n_k^M\right\}$. Individuals belong only to one club in each period of time, but they can move from one club to another at the beginning of every period.

For each population $\tau \in \{M, R\}$ we represent the set of mixed strategies by:

$$\Delta^\tau = \left\{ x \in R^{k_\tau} : \sum_{j=1}^{k_\tau} x_j = 1,\ x_j \geq 0,\ j = 1, ..., n_i \right\}$$

Note that, a profile distribution $x = (x_1, ..., x_{k_\tau}) \in \Delta^\tau$ can bee seen as the individual behavior of a player spending a part of his time, given by x_j, in the n_j-club, hence x represents the population state as the vector of individuals' share belonging to each club $i = 1, ..., k_\tau\ \forall\ \tau \in \{R, M\}$.

The normal form representation for our described game, is given by the next matrix payoff:

$R \diagdown M$	y_1	$\cdots$	y_{k_M}
x_1	a_{11}, b_{11}	$\cdots$	a_{1k_M}, b_{1k_M}
$\vdots$	$\vdots$	$\cdots$	$\vdots$
x_{k_R}	a_{k_R1}, b_{k_R1}	$\cdots$	$a_{k_Rk_M}, b_{k_Rk_M}$

(1)

where a_{ij} denotes the payoff of an $i-$strategist from population R playing against a $j-$strategist from population M. Conversely for b_{ij} from M to R.

The matching between individuals from different population is given in a random way. We use the notation:

$$E^R(n_i^R \mid y) = \sum_{j=1}^{k_M} a_{ij} y_j,\ \forall\ n_i^R \in S^R$$

to represent the $i-$strategist's expected payoff who belongs to the n_i-club from population R given that the fitness of strategists conform the clubs' distribution in y for the opposite population, M.

Analogously, the expected payoff of the $i-$strategist belonging to n_i-club from population M is given by:

$$E^M(n_i^M/x) = \sum_{j=1}^{k_R} b_{ij}x_j, \; \forall n_i^M \in S^M$$

where x is the clubs' distribution for the other population, R. Rational individuals follow the strategic profile that maximize the expected payoffs.

A more general case with n different populations can be considered by extending this model. In this case we consider a set of n populations indexed by $\tau = \{p_1, ..., p_n\}$ and each population splits in m_τ clubs. Consequently, if $y = (y^{p_1}, ..., y^{p_m})$ is the vector of distributions of the populations over its own clubs, i.e.: $y^{p_s} = (y_1^{p_s}, ...y_{m_s}^{p_s}) \in \Delta^s$, then $y_h^{p_s}$ represents the percentage of individuals of the population p_s belonging to the $n_h^{p_i}$ club, or equivalently, the percentage of individuals in the population p_s, following the h pure strategy or behavior, $1 \leq h \leq n_s$. So, the expected value for each strategist, in each population $p_i \in \tau$ will be denoted by:

$$E^{P_i}(n_h^{p_i}/y) = \sum_{1 \leq j_s \leq m_s \; \forall \; s \neq i} b^i h_i j_1...j_n y_{j_1}^{p_1}...y_{j_{i-1}}^{p_{i-1}} y_{j_{i+1}}^{p_{i+1}}...y_{j_n}^{p_n}$$

where $b^i h_i j_1...j_n$ denotes the payoff of an h pure strategist from the population p_i, given that the individuals from the population $p_s \neq p_i$ are playing according with j_s, $s \neq i$, pure strategy or behavior. However, without loss of generality, to simplify notation we shall work on the case of a two-population normal form games.

3 The asymmetric game and the definition of $\mathcal{ESS}$

Consider the above two-population normal form game:

$$G = \{(\tau \in \{R, M\}), \; S^\tau, \; (A = (a_{ij}), \; B = (b_{ij}))\} \tag{2}$$

where each population splits into clubs denoted by $n_i^\tau, \;\; \forall \; \tau = \{R, M\}$ and $i = 1, ..., k_\tau$. Hence:

- The population of residents is the set: $R = \bigcup_{i=1}^{k_R} n_i^R$, and $\; \forall \; h \neq j \; n_h^R \bigcap n_j^R = \emptyset$.
- The population of migrants is the set: $M = \bigcup_{i=1}^{k_M} n_i^M$, and $\; \forall \; h \neq j \; n_h^M \bigcap n_j^M = \emptyset$.

Let $p \in \Delta^\tau$ be the profile distribution of individuals' behavior from population R, in a given period of time t_0, and that in the same time, the profile distribution of individuals' behavior in population M is $q \in \Delta^\tau$. Assume that in a post-period of time $t_1 > t_0$ a small mutation affects the individuals' behavior from population M. Hence, the profile distribution from population M after the mutation, is denoted by the offspring:

$$q_\epsilon = ((1 - \epsilon)q + \epsilon\bar{q},$$

which is called the fitness of the post-entry population. Analogously, the profile distribution from population R after suffering a small mutation is:

$$p_\epsilon = ((1-\epsilon)p + \epsilon\bar{p}.$$

Now, we can state the next definition:

Definition 1 *Let $(p^*, q^*) \in \Delta^R \times \Delta^M$ be a profile of mixed strategies. We say that the profile (p^*, q^*) is an $\mathcal{ESS}$ for an asymmetric two-population normal form game, if for each $(\bar{p}, \bar{q}) \neq (p^*, q^*) \in \Delta^R \times \Delta^M$ there exists $\bar{\epsilon}$ such that:*

$$\begin{aligned} &1) \quad E^R(p^*/q^*_\epsilon) > E^R(\bar{p}/q^*_\epsilon) \ \textit{and} \\ &2) \quad E^M(q^*/p^*_\epsilon) > E^M(\bar{q}/p^*_\epsilon), \end{aligned} \tag{3}$$

for all ϵ, $0 < \epsilon \leq \bar{\epsilon}$, where $p^*_\epsilon = (1-\epsilon)p^* + \epsilon\bar{p}$ and $q^*_\epsilon = (1-\epsilon)q^* + \epsilon\bar{q}$, are the respective post-entry populations.

So, individuals' behavior who adopt an $\mathcal{ESS}$ brings more offspring (with higher fitness) than the mutant individuals' behavior from the post-entry population.

Definition 1 can be extended to the case of multipopulation models. For such cases we consider $x = (x^{p_1}, ..., x^{p_m})$ such that $x^{p_i} \in \Delta^i, i = 1, ..., m$ is a distribution of probability over the set of clubs or pure strategies, for each population. So, x^* is an $\mathcal{ESS}$ if and only if for each $\bar{x} \neq x^*$, there exist an $\epsilon_{\bar{q}} > 0$ such that the following inequalities hold:

$$E^{p_i}(x^{*p_i}/x^*_\epsilon) > E^{p_i}(\bar{x}^{p_i}/x^*_\epsilon), \quad \forall\, p_i \in \tau, \text{ and } 0 < \epsilon < \epsilon_{\bar{x}}$$

where $x^*_\epsilon = (1-\epsilon)x^* + \epsilon\bar{x}$.

The following theorem characterizes the $\mathcal{ESS}$ in terms of Nash equilibria (see, for instance, Swinkels J., 1992).

Proposition 1 *A profile x is $\mathcal{ESS}$ if and only if x is a strict Nash equilibrium.*

The evolutive properties of the $\mathcal{ESS}$ and its relationship with the set of Nash equilibria and the stationary states ($\mathcal{SS}$) of the replicator dynamics for the case of symmetric games are well known (see Hofbauer and Sigmund (1998); Weibull (1995)). Then, with the purpose of analyzing the dynamical properties of the $\mathcal{ESS}$, let us introduce the symmetric (one-population) version for the asymmetric two-population game, G.

4 The symmetrized game, the $\mathcal{NE}$ and $\mathcal{ESS}$

Consider, the asymmetric two-population normal form game G defined by (2), where each population splits into clubs $n_1^R, ..., n_{k_R}^R$ and $n_1^M, ..., m_{k_M}^M$ and the payoffs matrix are A and B, respectively. Its corresponding symmetrized one-population game is defined as:

Definition 2 *Let G be an asymmetric game defined by (2), consider:*

1. *The big population: $P = R \cup M$.*
2. *Individuals from the big population P face their own population.*
3. *Let $N = \left\{ n_1^R, ..., n_{k_R}^R, n_1^M, ..., m_{k_M}^M \right\}$ be the set of pure strategy for P.*
4. *The matrix payoff for the big population P is:*

$$\Pi = \left[\begin{array}{cc} 0 & A \\ B^T & 0 \end{array} \right] \tag{4}$$

Hence, the numbered list item 1-4 characterizes the symmetrized game version $G^s = \{P,\ N,\ \Pi\}$ of the asymmetric game G.

Much of the work on evolution has been studied for the case of a single homogeneous population playing a symmetric game like G^s. For this reason, our interest is to use the well know properties of the symmetric games, and so to obtain the main characteristics of the $\mathcal{ESS}$ in an asymmetric (original) games. This will be doing by using the symmetrized version of the asymmetric game. Then, for each asymmetric two-population game G, there exists a corresponding symmetric version as defined by G^s. It is worth to note that, these two versions are not equivalent in several aspects,[1] but as we shall show every Nash equilibrium of the asymmetric game is a Nash equilibrium of the symmetric version. Hence, our purpose is to characterize the main dynamics properties of the $\mathcal{ESS}$, and to do this we do not need a full equivalence between these two versions.

Let us consider the strategic profile $(p,\ q) \in \Delta^p \times \Delta^q$, the profile distribution $x = (x_1, ..., x_{k_R + k_M})$ verifying the following identities:

$$x_i = \begin{cases} p_i \frac{|R|}{|R|+|M|} & \text{if } 1 \leq i \leq k_R \\ \\ q_i \frac{|M|}{|R|+|M|} & \text{if } k_R < i \leq k_R + k_M \end{cases} \tag{5}$$

(where by $|\cdot|$ we denote the cardinality on the sets R and M defining the corresponding mixed strategy for the symmetric version G^s.

Proposition 2 *For each strategic profile $(p,\ q) \in \Delta^R \times \Delta^M$, there exists a mixed strategy $x \in \Delta^P$ of the corresponding one-population game, and reciprocally.*

Proof. Let $(p,\ q) \in \Delta^R \times \Delta^M$ be a strategic profile for the asymmetric game. Consider $x \in \Delta^P$ given by the expression (5), i.e. $x = \left(\frac{|R|}{|M|+|R|} p,\ \frac{|M|}{|M|+|R|} q \right)$. Thus, x is a mixed strategy for the symmetric game. To see the reciprocal, suppose that $x \in \Delta^P$ and consider the above equalities

[1] For instance, for expected payoffs not invariant with respect to positive affine transformations, i.e. substracting a sufficiently large constant from all payoffs in the asymmetric game; then the equilibria of the asymmetric game are unchanged, but in the symmetric version all symmetric strategy combinations become equilibria.

but in the opposite sense, since $x_i = \frac{|R_i|}{|R|+|M|}$ if $1 \leq i \leq k_R$ and $x_i = \frac{|M_i|}{|R|+|M|}$ if $k_R < i \leq k_R + k_M$ where $|R_i|$ represents the cardinality of individuals in the n_i^R club, $i = 1, ..., k_R$, analogously for $|M_i|$, $i = 1, ..., k_M$. So, $p_i = \frac{|R|+|M|}{|R|} x_i$, $i = 1, ..., k_R$ and $q_i = \frac{|R|+|M|}{|M|} x_i$, $i = k_R + 1, ..., k_R + k_M$. ■

Now, let us denote by $B_\tau(z)$ the set of best replies for the population $\tau \in \{M, R\}$, where the profile distribution over the clubs in the opposite population $\tau' \neq \tau$ is given by z, and $\tau' \in \{M, R\}$.

The following propositions offer an insight about the relationship between the set of $\mathcal{NE}$ and the set of $\mathcal{ESS}$ for asymmetric games and their respective symmetric versions.

Proposition 3 *If the strategic profile (p^*, q^*) is a $\mathcal{NE}$ of the original asymmetric two-population game, then the corresponding x^* defined by the expression (5) is the symmetric $\mathcal{NE}$ in the corresponding symmetric version.*

Proof. Suppose that the profile (p^*, q^*) is a $\mathcal{NE}$ of the asymmetric two-population game. Let $x^* = (x_1^*, ..., x_{k_M+k_R}^*)$ be the corresponding strategy in the corresponding symmetrized one-population game (Definition 2). Then, note that $p^* \in B_R(q^*)$ and $q^* \in B_M(p^*)$ implies that $x^* P x^* \geq y P x^*$ for all $y \in \Delta^P$. To see this, consider that for each $y \in \Delta^P$ the following relations:

$$p_i = \frac{|R|+|M|}{|R|} y_i \qquad \text{if } 1 \leq i \leq k_R,$$

$$q_{i-n_k^R} = \frac{|R|+|M|}{|M|} y_i \quad \text{if } (k_R + 1) \leq i \leq k_R + k_M$$

thus, $p = (p_1, ...p_{k_R}) \in \Delta^R$ and $q = (q_1, ..., q_{k_M}) \in \Delta^M$.

$$yPx^* = \frac{|M||R|}{(|M| + |R|)^2} \left(qB^T p^* + p^* A q \right) \leq \frac{|M||R|}{(|M| + |R|)^2} \left(q^* B^T p^* + p^* A q^* \right) = x^* P x^*.$$

■

Proposition 4 *If the profile (p^*, q^*) a strict Nash equilibrium for the asymmetric two population game, then the corresponding x^* is a strict Nash equilibrium for the symmetric version.*

Proof. Let (p^*, q^*) be a strict Nash equilibrium for the asymmetric two population game and let x^* the corresponding profile for the symmetric version. Assume that there there exist $y \neq x^* \in \Delta^p$, such that $y \Pi x^* = x^* \Pi x^*$ then, using proposition 2, there exist $p \neq p^*$ such that $pAq^* \geq p^* A q^*$ or, there exist $q \neq q^*$ such that $p^* B q \geq p^* B q$, contradicting our assumption. ■

Proposition 5 *If the profile (p^*, q^*) is an $\mathcal{ESS}$ for the asymmetric two-population game, then the corresponding x^* is an $\mathcal{ESS}$ for the symmetric version.*

Proof. Let (p^*, q^*) be an $\mathcal{ESS}$, then by proposition (1) is a strict Nash equilibrium. Now, from proposition (4) the corresponding strategy x^* is a strict Nash equilibrium for the symmetric version, and then $\mathcal{ESS}$. ■

Remark 1 *It is straightforward to see that the reciprocal of this proposition does not hold.*

Recall that, the symmetric version and the original asymmetric game are not fully equivalent, but our main interest is to characterize the dynamical properties of the solutions for asymmetric games. So, as long as the solutions of an asymmetric game are still solutions of the symmetric version, we can use this version with this purpose.

5 The dynamics of the model

Our main point in this section is to analyze the evolutionary dynamics of two populations engaged in an asymmetric environment when the inhabitants follow a rational behavior. The symmetric version of the asymmetric game allows to characterize the main dynamical properties of the asymmetric $\mathcal{ESS}$, because these properties are well known in this case.

Consider the asymmetric two-population normal form game, G, represented by the list numbered (2). Let us denote the following:

1. Let $n_i^\tau(t)$ be the number of individuals at time t belonging to the $i-$club in the population τ.
2. Let $p_i(t)$ the share of individuals in the $i-$club from the population R and analogously $q_i(t)$ the share of individuals in the $i-$club from the population M, at time t. Hence,

$$p_i(t) = \frac{n_i^R}{|R|}$$

and

$$q_i(t) = \frac{n_i^M}{|M|}$$

3. Hence, $(p(t),\, q(t))$ is the profile distribution (or population state) at time t from each population R and M respectively. Then, $p(t) \in \Delta^R$ and analogously, $q(t) \in \Delta^M$.

The members of the $i-th$ club from population τ, are called $i-$strategists in the population $\tau \in \{R, M\}$. Rational individuals choose strategies to maximize their expected payoffs. Certainly this set of maximizing strategies depends on the strategies displayed by the other population. Let $z_0 = (p_0, q_0)$ be the strategic profile at time $t = 0$ for the asymmetric two-population game G. According to the rationality it follows that:

$$\begin{aligned} \dot{p}_i &= ((e_i^R - p)Aq)p_i,\ i = 1, ..., k_R \\ \dot{q}_i &= ((e_i^M - q)B^T p))q_i,\ i = 1, ..., k_M, \end{aligned} \tag{6}$$

where e_i^R is the $i-$canonical vector in $\Re^R$ and e_i^M is the canonical $i-$th vector in the $\Re^M$. System (6) represents the clubs' evolution for each population. For the system (6), a solution of the form: $\xi(t, z_0) = (\xi_1(t, z_0), \xi_2(t, z_0))$ represents the evolution of the population states with initial state given by z_0.

From the system (6) it is straightforward to see that in each time t the club of the $i-$strategists in each population increases if and only if the expected payoff of the $i-$strategy is greater than the average payoff, and reciprocally.

Note that, for each pair $(p(t), q(t))$ in G, there exists a corresponding mixed strategy $x(t)$ in the symmetric version G^s given by the equivalence (5).

Then, the dynamical system (6) has a corresponding dynamical system, namely the replicator dynamics, (see Taylor and Jonker, 1978) of the symmetric one-population game given by:

$$\dot{x}_i = ((e_i - x)Px)x_i = \begin{cases} ((e_i^M - q)B^T p)q_i & \text{if } 1 \leq i \leq k_R \\ \\ ((e_i^R - p)Aq)p_i & \text{if } (k_R + 1) \leq i \leq k_R + k_M \end{cases} \tag{7}$$

where e_i is the $i-th$ canonical vector in $\Re^{k_R+k_M}$.

To study the relationship between $\mathcal{NE}$, $\mathcal{ESS}$ and $\mathcal{SS}$ for the system (6) of the asymmetric game, G, can be done by means of analyzing the dynamics corresponding to the symmetric version game, G^s, from its replicator dynamics (7).

The following propositions are straightforward from the respective definitions:

Proposition 6 *If a pair $(\bar{p}, \bar{q})$ is a stationary state for the system (6) then the corresponding $\bar{x}$ is a stationary state for the dynamical system (7).*

Proposition 7 *Every strictly positive stationary state of the dynamical system (6) is a $\mathcal{NE}$ for the corresponding asymmetric two-population game.*

Proposition 8 *Every $\mathcal{NE}$ of an asymmetric two-population game is a stationary state for its corresponding dynamical system given by (6).*

Hence, we can conclude that the set of $\mathcal{NE}$ of an asymmetric two-population game is a subset of the set $\mathcal{SS}$ corresponding to the dynamical system (6).

Corollary 1 *Every $\mathcal{NE}$ of a two-population game is a stationary state for the corresponding dynamical system (7).*

Proof. By propositions (8) and (6) the corollary follows. ■

6 Evolutionarily stable strategies and Liapunov's stability

Hofbauer and Sigmund (1988) pointed out a proof that for non-homogeneous asymmetric two population games, interior points cannot be asymptotically stable steady states of the replicator dynamics. On the other hand, we know that the concepts of strict Nash equilibrium and $\mathcal{ESS}$ are equivalents in symmetric games. We shall prove using the symmetric version of an asymmetric game, that every $\mathcal{ESS}$ is an asymptotically stable steady state of the replicator dynamics.

Let us give a proper analysis from our model. We denote by $\mathcal{AS}$ the set of asymptotically stable steady states.

From the definition of $\mathcal{ESS}$ (Definition 1) in the case of an asymmetric two-population game and from the propositions (3), (6) and (8), and using the well known relations between $\mathcal{ESS}$, $\mathcal{NE}$ and $\mathcal{SS}$ for the symmetric cases (see Weibull (1995)), the following relationship holds for every asymmetric two-population game:

$$\mathcal{ESS} \subseteq \mathcal{AS}, \tag{8}$$

and

$$\mathcal{NE} \subseteq \mathcal{SS}. \tag{9}$$

Proposition 9 *For an asymmetric two-population game it follows that if (p^*, q^*) is an asymptotically stable steady state corresponding to the dynamical system (6), then it is a $\mathcal{NE}$.*

Proof. If $(p^*, q^*) \in AS$ for the dynamical system (6) then it is stationary state. If $p^* >> 0$ and $q^* >> 0$ then from Proposition (7) it follows that (p^*, q^*) is a $\mathcal{NE}$ for the asymmetric game. Now we consider the case where some strategy is absent in p^* or in q^*. Without lost of generality assume that $p_j^* = 0$. This means that actually there are not individuals in the n_j^R club. Suppose now that (p^*, q^*) is not a $\mathcal{NE}$. Then there exists some pure strategy $j \notin supp(p^*)$ such that $E^R(e_j^R/q^*) = e_j^R Aq^* > p^* Aq^* = E^R(p^*/q^*)$. Assume that a perturbation affects the distribution p^* and that in the population R some $j-$strategist appear. So, the post-entry population in time t is $p_\epsilon(t) = (1-\epsilon(t))p^* + \epsilon(t)e_j^R$. Substituting in the $j-th$ differential equation in the system (6) we obtain:

$$\dot{p}_{\epsilon j} = \dot{\epsilon} = [(e_j^R - p_\epsilon)Aq^*]\epsilon. \tag{10}$$

Let us now define $F(\epsilon) = (e_j^R - p_\epsilon)Aq^*$. Note that $F(0) = (e_j^R - p^*)Aq^*$ and $F'(0) = (p^* - e_j^R)Aq^*$. So, the Taylor polynomial is $F(\epsilon) = F(0) + F'(0)\epsilon + 0(\epsilon^2)$. Now considering (in equation (10)), the first order approximation it follows that:

$$\dot{\epsilon} = [(e_j^R - p^*)Aq^*]\epsilon. \tag{11}$$

So, in the population R the members in the n_j^R club increase, contradicting our claim that (p^*, q^*) is an asymptotically stable steady state with $n_j^R = 0$. ∎

We turn now to the connection between $\mathcal{ESS}$ and the replicator dynamics in an asymmetric game. We will use the following proposition, see Taylor and Jonker 1978:

Proposition 10 *For symmetric homogeneous population game every $\mathcal{ESS}$ is an asymptotically stable steady state of the replicator dynamics.*

The following corollary holds:

Corollary 2 *For the asymmetric two-population game we obtain the following chain of inclusions:*

$$\mathcal{ESS} \subseteq \mathcal{AS} \subseteq \mathcal{NE} \subseteq \mathcal{SS}.$$

Proof. Let (p^*, q^*) be an $\mathcal{ESS}$ for an asymmetric game and let x^* be the corresponding strategic profile in its symmetric version. So, by Proposition 1, it follows that (p^*, q^*) is a strict Nash equilibrium. By Proposition 4, it follows that the symmetric strategic profile of every strict Nash equilibrium of an asymmetric game is an strict $\mathcal{NE}$. Then x^* is an strict Nash equilibrium for the symmetric version, and then x^* is a $\mathcal{ESS}$. Now, by Proposition 10, it follows that x^* is an asymptotically stable steady state of the replicator dynamics. Then, $(p^* q^*)$ is an asymptotically stable steady state for the asymmetric version, so being a $\mathcal{NE}$. ∎

Bomze, I. (1986) shows that every asymptotically stable steady state in the homogeneous population replicator dynamic corresponds to a Nash equilibrium that is trembling hand. However, from our model using the symmetric version (Definition 2) of a non-homogeneous asymmetric $n-$population the following proposition follows:

Proposition 11 *Every $\mathcal{ESS}$ of a non-homogeneous asymmetric $n-$population game is trembling hand and isolate.*

Proof. Let (p^*, q^*) be an $\mathcal{ESS}$ for an asymmetric game and let x^* be the corresponding strategic profile in its symmetric version. By Corollary 2 it follows that every $\mathcal{ESS}$ is asymptotically stable for the symmetric version. So, x^* is asymptotically stable steady state for the symmetric version. Now, taking account the above result due to Bomze (1986), it follows that x^* is trembling hand and isolate equilibrium, and so (p^*, q^*) verifies this property in the original asymmetric game. ■

7 Concluding remarks

In this paper, we extended the definition of evolutionarily stable strategies ($\mathcal{ESS}$) of symmetric games to asymmetric two-population games. We made this by taking as the strategy space for the symmetrized game the union of strategies from the two-population asymmetric game and assigning zero payoff to all strategy combinations that belong to the same player position in the asymmetric game. With this symmetrized game, we show again some well-known relationships between static and dynamic stability notions.

Hence, evolutionary dynamics in a two-population asymmetric game can be analyzed using the well known properties of the replicator dynamic corresponding to the symmetric version of this game. This fact, may have interest for economic theory, or social analysis, where asymmetric games are useful to analyze the behavior of two populations engaged in non-cooperative games such as, buyers and suppliers, firms and workers or residents and migrants populations interacting in a given country or economy. The reference of the symmetric version from an asymmetric two-population games allow us to generalize the results given in the existing literature.

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