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Stable Bernoulli diffeomorphisms in dimension three

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Programa de Posgrado en Matemática
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Dedicado a mi hijo Emmanuel,
mi compañera de vida Melanie,
mi madre Elba y mi hermano
Miguel.

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RESUMEN

Sea M una variedad compacta y m un volumen en M. Denotamos $\mathrm{Diff}^r_m(M)$ el conjunto de los difeomorfismos C^r-conservativos en M. Una foliacion es minimal si toda hoja es densa en M. En esta tesis probaremos que si M tiene dimensión tres, entonces genéricamente en $\mathrm{Diff}^1_m(M^3)$, la existencia de una foliación invariante, minimal y expansora implica estabilidad Bernoulli.

También damos condiciones para garantizar la persistencia de una foliación minimal expansora de una variedad M de cualquier dimensión.

Palabras claves:
Estabilidad Ergódica, Estabilidad Bernoulli, Foliación minimal, No-uniformemente hiperbólico.

ABSTRACT

Let M be a smooth compact manifold and let m be a smooth volume measure. We denote by $\mathrm{Diff}^r_m(M)$ the set of C^r-conservative diffeomorphisms. A foliation is minimal if every leaf is dense in M. In this work, we prove that is M has dimension three, then generically in $\mathrm{Diff}^1_m(M^3)$, the existence of a minimal expanding invariant foliation implies stable Bernoulliness.

We also find conditions under which a minimal expanding foliation persists and is minimal for a manifold M of any dimension.

Keywords:
Stable ergodicity, Stable Bernoulliness, Minimal foliation, Non-uniformly hyperbolic.

Contents

Chapter 1

Introduction

1.1 Historical context and presentation of the results

Let M be a smooth compact manifold and let m be a smooth volume measure. A diffeomorphism $f : M \to M$ is *ergodic* if the Birkhoff's limits

$$\tilde{\varphi}(x) = lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi\circ f^j(x)$$

are constants for m-almost every $x \in M$ for all $\varphi : M \to \mathbb{R}$ continuous function.

In 1939 E. Hopf [Hop39] proved that the geodesic flow of a surface with negative sectional curvature is ergodic with respect to the Liouville measure. This result, which is now called the Hopf argument, was generalized by D. Anosov [Ano69], who in the late sixties showed that conservative C^2-Anosov diffeomorphisms (and flows) are C^1 stably ergodic. That is, given a C^2 conservative diffeomorphism there exists $\mathcal{U}$ a C^1 neighborhood of it such that every conservative C^2 element in $\mathcal{U}$ is ergodic. Similarly for the case of flows. The crucial tool for to do it was the absolute continuity of stable and unstable due to D. Anosov and Ya. Sinai [AS67]. It is not known yet whether C^1-Anosov diffeomorphism are ergodic.
Until 1993, Anosov diffeomorphisms were the only known conservative examples of stably ergodic diffeomorphisms, but Grayson, Pugh and Shub showed

that the time-one map of the geodesic flow of a surface of negative curvature is stably ergodic [GPS94].

In 1995, Pugh and Shub conjectured that partially hyperbolic diffeomorphism are generically stably ergodic, i.e., in a certain way, a little hyperbolicity goes a long way toward guaranteeing stable ergodicity. The Pugh-Shub conjecture was proposed in the International Congress on Dynamical Systems, held in Montevideo in 1995, in the memory of Ricardo Mañé [PS96]. This conjecture has been very active and it continues to be.

The partially hyperbolic diffeomorphisms are those diffeomorphisms such that the tangent bundle TM splits in three Df-invariant subbundles $E^u \oplus E^c \oplus E^s$, where E^u is expanding (called the unstable bundle), E^s is contracting (stable bundle) and E^c is intermediate (central bundle). See [HHTU11] for the precise definition.

The Pugh-Shub conjecture was proved by F. Rodríguez Hertz, J. Rodríguez Hertz and R. Ures [RHRHU08] when the central subbundle is one dimensional and also, for the C^1 topology, when the central subbundle is two dimensional by the same authors and A. Tahzibi [HHTU11].

In this context, it was natural to ask if the stable ergodicity implies partial hyperbolicity. A. Tahzibi in his Ph.D. Thesis [Tah04] gave an example of a stably ergodic diffeomorphims which is not partially hyperbolic. The map, a diffeomorphism of T^4, was introduced before by Bonatti-Viana in [BV00]. Even though the map is not partially hyperbolic it has a dominated splitting, this is: the tangent bundle over M splits into two subbundles $TM = E \oplus F$ such that given any $x \in M$, any unitary vectors $v_E \in E(x)$ and $v_F \in F(x)$:

$$\| Df^N(x)(v_E) \| \leq \frac{1}{2} \| Df^N(x)(v_F) \|$$

for some $N > 0$ independent of x.

Recently the Pugh-Shub conjecture was proved in the C^1-topology for any dimension of central subbundle by A. Avila, S. Crovisier y A. Wilkinson [ACW17].

The main objective in this work will be to show that "a little hyperbolicity goes a long way toward guaranteeing stable ergodicity". To state our main results, let us recall some definitions.

A foliation W is *minimal* if every leaf $W(x)$ of W is dense in M. An f-invariant foliation W is *contracting* if $\|Df(x)|_{TW}\| < 1$ for every $x \in M$. An f-invariant foliation is *expanding* if it is contracting for f^{-1}.

We say that a diffeomorphism $f \in \mathrm{Diff}^1_m(M)$ is *stably ergodic* if there exists a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^1_m(M)$ of f such that all $g \in \mathcal{U}(f) \cap \mathrm{Diff}^2_m(M)$ are ergodic. Note that our definition of stable ergodicity does not imply that the diffeomorphism itself is ergodic in case it is only C^1. However, if $f \in \mathrm{Diff}^2_m(M)$ then f will be both stably ergodic and ergodic.

A diffeomorphism $f \in \mathrm{Diff}^1_m(M)$ is *non-uniformly hyperbolic* if all its Lyapunov exponents are non-zero m-almost everywhere, that is if for m-almost every x, and every unit vector $v \in T_xM$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| \neq 0.$$

A diffeomorphism $f \in \mathrm{Diff}^1_m(M)$ is *stably non-uniformly hyperbolic* if there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ such that all diffeomorphisms g in $\mathcal{U} \cap \mathrm{Diff}^2_m(M)$ are non-uniformly hyperbolic. A diffeomorphism f in $\mathrm{Diff}^1_m(M)$ is *Bernoulli* if it is metrically isomorphic to a Bernoulli shift [1]. The diffeomorphism f is *stably Bernoulli* if there exists a neighborhood $\mathcal{U}$ in $\mathrm{Diff}^1_m(M)$ such that all diffeomorphisms g in $\mathcal{U} \cap \mathrm{Diff}^2_m(M)$ are Bernoulli.

The next results are joint work with Jana Rodríguez Hertz (SUSTech). Let M be a 3-dimensional manifold.

Theorem 1.1.1 *[G. Núñez, J. Rodríguez Hertz] There exists a residual set $\mathcal{R}$ in $\mathrm{Diff}^1_m(M^3)$ such that for $f \in \mathcal{R}$, if there exists a minimal expanding or contracting f-invariant foliation, then f is stably Bernoulli and stably non-uniformly hyperbolic.*

We remark that the foliation above does not have to be the *most* expanding or contracting invariant foliation: it could be an intermediate foliation. Also,

[1] i.e. there exist measurable functions $h : M \to \Sigma$ and $k : \Sigma \to M$ such that $h \circ k = id_\Sigma$, μ_Σ a.e.p., $k \circ h = id_M$, m a.e.p., $m(h^{-1}(B)) = \mu_\Sigma(B)$, for all $B \subset \Sigma$ measurable and $f \circ h = h \circ T$ m-a.e.p., where T is the shift transformation.

that we do not require *a priori* that there be a dominated splitting, though *a fortiori* it will be the case for the generic diffeomorphism.

In the conclusion of Theorem 1.1.1, f will be both stably ergodic and ergodic. The same comment applies for stable non-uniform hyperbolicity. In the Theorem above, f will be also non-uniformly hyperbolic.

An f-invariant foliation W is *stably minimal* if there exists a C^1-neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}^1_m(M)$ such that

1. For each $g \in \mathcal{U}$ there exists a g-invariant foliation W_g such that the fiber bundle $g \mapsto TW_g$ varies continuously on $\mathcal{U}(f)$
2. W_g is minimal for all $g \in \mathcal{U}(f) \cap \mathrm{Diff}^2_m(M)$

Theorem 1.1.2 *[G. Núñez, J. Rodríguez Hertz] There exists a residual set $\mathcal{R}$ in $\mathrm{Diff}^1_m(M^3)$ such that for $f \in \mathcal{R}$ if there exists a minimal expanding or contracting f-invariant foliation W, then either W is stably minimal or else W is the strongest foliation of an Anosov diffeomorphism.*

The proof is based in two main tools. One is the *Pesin homoclinic classes* associated to a hyperbolic periodic point, introduced in [HHTU11] which we define in detail in the chapter 2 - section 2.7, and the other is the genericity result obtained by [Her12] in dimension 3 and by [ACW16] in any dimension, which we also state below.
For any dimension we have the next theorem. It is not a generic theorem.

Theorem 1.1.3 *[G. Núñez, J. Rodríguez Hertz] Let $f \in \mathrm{Diff}^1_m(M)$ and W an f-invariant expanding minimal foliation such that:*

1. *There exists a Df-invariant sub-bundle of TM, F such that the splitting $TM = F \oplus_< TW$ is dominated.*
2. *there exists a hyperbolic periodic point p_f with unstable index $u = \dim W$.*
3. *there exists a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^1_m(M)$ such that for all $g \in \mathcal{U}(f) \cap \mathrm{Diff}^2_m(M)$, $m(\mathrm{Phc}^-(p_g)) > 0$, where p_g is the analytic continuation of the periodic point p_f.* [1]

Then W is stably minimal and f is stably Bernoulli.

[1] $\mathrm{Phc}^-(p_g)$ is the set of x in M whose Pesin stable set $W^-(x)$ intersect $W^+(o(p_g))$ in a transverse way.

Also, we have a criterion that guaranteeing the minimality of a expanding or contracting foliation. In this result we use the ideas and arguments showed in [BDU02].

Theorem 1.1.4 (Minimality Criterion) *Given a diffeomorphism* $f \in \mathrm{Diff}^1_m(M)$, *an expanding* f*-invariant foliation* W^u, *and a hyperbolic periodic point* $p \in M$ *such that*

1. *the unstable index of* p, $u(p)$ *equals* $\dim W^u$
2. $\mathrm{Phc}^u(p) = M$
3. $\overline{W^u(p)} = M$

Then W^u *is a minimal foliation.*

The main theorems 1.1.1 and 1.1.2 for high dimension are more delicate. This is a work in progress with J. Rodríguez Hertz.

Chapter 2

Preliminaries

2.1 Dominated Splitting

In this section, we will define the concept of dominated splitting and we will show some elementary properties. The refer the reader to [BDV05], [CP15] and [Sam14] for further information on the topic.
Let V and W be two normed vector spaces (over the same field) and let $T : V \to W$ be a continuous linear map. We define the *norm* of T as

$$\| T \| = \sup \left\{ \frac{\| T(v) \|}{\| v \|} : v \neq 0 \right\}$$

and the *minimal norm* or *co-norm* as:

$$m(T) = \inf \left\{ \frac{\| T(v) \|}{\| v \|} : v \neq 0 \right\}$$

Clearly, we have

$$m(T) \| v \| \leq \| T(v) \| \leq \| T \| \| v \|$$

and when T is invertible

$$m(T) = \| T^{-1} \|^{-1}$$

Also, if $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are continuous linear maps (where V_1, V_2 and V_3 are normed vector spaces over the same field) then

$$\| S \circ T \| \leq \| S \| \| T \| \text{ and } m(S \circ T) \geq m(S)m(T).$$

Let $f : M \to M$ be a diffeomorphism on a closed manifold M and Λ be any f-invariant set. A Df-invariant splitting $T_\Lambda M = E \oplus F$ of the tangent bundle is *dominated*, and denote it by $T_\Lambda M = E \oplus_< F$ if there is $N \geq 1$ such that given any $x \in \Lambda$, any unitary vectors $v_E \in E(x)$ and $v_F \in F(x)$, we have:

$$\| Df^N(x)(v_E) \| \leq \frac{1}{2} \| Df^N(x)(v_F) \| \tag{2.1}$$

More generally, a Df-invariant splitting $T_\Lambda M = E_1 \oplus E_2 \oplus \ldots \oplus E_n$ of the tangent bundle is dominated if for all $k \in \{1, \ldots, n-1\}$ we have the splitting

$$(E_1 \oplus \ldots \oplus E_k) \oplus (E_{k+1} \oplus \ldots \oplus E_n)$$

is dominated. In this case we write $E_1 \oplus_< \ldots \oplus_< E_n$.
If we have a dominated splitting $E_1 \oplus_< \ldots \oplus_< E_n$ always exists a unique *finest dominated splitting* $F_1 \oplus_< \ldots \oplus_< F_k$ over Λ (see [BDP03], Proposition 4.11) characterized by the following property: given any dominated splitting $E \oplus_< F$ over Λ then there is some $l \in \{1, 2, \ldots, k-1\}$ such that

$$E = F_1 \oplus_< \ldots \oplus_< F_l, \text{ and } F = F_{l+1} \oplus_< \ldots \oplus_< F_k$$

Remark 2.1.1 *The condition (2.1) clearly is equivalent to the condition:*

$$\frac{\| Df^N(x)(v_E) \|}{\| v_E \|} \leq \frac{1}{2} \frac{\| Df^N(x)(v_F) \|}{\| v_F \|}$$

for every $v_E \in E(x) \setminus \{0\}$ *and* $v_F \in F(x) \setminus \{0\}$.
Also, it is equivalent to the condition:

$$\| \left. Df^N \right|_{E(x)} \| \leq \frac{1}{2} m \left(\left. Df^N \right|_{F(x)} \right)$$

The next proposition give us a equivalent definition for the dominated splitting.

Proposition 2.1.1 *Let* $f : M \to M$ *be a diffeomorphism on a closed manifold* M *and* Λ *be any* f*-invariant set. Then the splitting* $T_\Lambda M = E \oplus F$ *is dominated if and only if there exist* $C > 0$ *and* $\lambda \in (0, 1)$ *such that given any* $x \in \Lambda$*, any unitary vectors* $v_E \in E(x)$ *and* $v_F \in F(x)$*, we have:*

$$\| Df^n(x)(v_E) \| \leq C\lambda^n \| Df^n(x)(v_F) \|, \text{ for all } n \geq 1$$

Proof. The converse implication follows immediately, because the existence of the constants that satisfy $\| Df^n(x)(v_E) \| \leq C\lambda^n \| Df^n(x)(v_F) \|$ imply the definition of dominated splitting. Let's see the direct implication.
Write $n = kN + r$, with $0 \leq r < N$. Then,

$$\| Df^n|_{E(x)} \| \leq \| Df^r|_{E(f^{kN}(x))} \| \| Df^N|_{E(x)} \|^k$$

Writing $A_r(x) = \| Df^r|_{E(f^{kN}(x))} \|$ then, by the domination, we have:

$$\| Df^n|_{E(x)} \| \leq A_r(x) \left(\frac{1}{2}\right)^k m\left(Df^N|_{F(x)}\right)^k$$

and then

$$\| Df^n|_{E(x)} \| \leq \frac{A_r(x)}{B_r(x)} \left(\frac{1}{2}\right)^k m\left(Df^n|_{F(x)}\right)$$

where $B_r(x) = m\left(Df^r|_{F(f^{kN}(x))}\right)$.
Let $\tilde{C} = \sup\left\{\frac{A_r(x)}{B_r(x)} : x \in \Lambda,\, 0 \leq r < N\right\}$, then

$$\| Df^n|_{E(x)} \| \leq \tilde{C} \left(\frac{1}{2}\right)^k m\left(Df^n|_{F(x)}\right)$$

Taking $\lambda = \left(\frac{1}{2}\right)^{1/N} \in (0,1)$ and $C = \frac{\tilde{C}}{\lambda^r} > 0$, we have:

$$\| Df^n|_{E(x)} \| \leq C\lambda^n m\left(Df^n|_{F(x)}\right)$$

This complete the proof. $\square$

Given $U \subset M$ with a splitting $T_UM = \tilde{E} \oplus \tilde{F}$ into continuous subbundles (not necessarily invariant) and $\alpha \in (0,1)$ we can define a *cone-field* in U as:

$$\mathcal{C}^F_\alpha(x) = \{v = v_{\tilde{E}} + v_{\tilde{F}} \in T_xM : \| v_{\tilde{E}} \| \leq \alpha \| v_{\tilde{F}} \|\}$$

for each $x \in U$.
Also, we define for each $x \in U$ the complementary cone:

$$\mathcal{C}^E_\alpha(x) = \{v = v_{\tilde{E}} + v_{\tilde{F}} \in T_xM : \| v_{\tilde{E}} \| \geq \alpha \| v_{\tilde{F}} \|\}$$

Clearly we have $F_x \subset \mathcal{C}^F_\alpha(x)$ for every $\alpha \in (0,1)$ and if $\alpha < \beta$ then $\mathcal{C}^F_\alpha(x) \subset$

$\mathcal{C}^F_\beta(x)$. Also, we have something similar for $\mathcal{C}^E_\alpha(x)$

Remark 2.1.2 *If the splitting $T_\Lambda M = E \oplus F$ is dominated then by the proposition 2.1.1 there exist $C > 0$ and $\lambda \in (0,1)$ such that for every $x \in \Lambda$ and $n \in \mathbb{N}$ we have:*

$$Df^n(x)(\mathcal{C}^F_\alpha(x)) \subset \mathcal{C}^F_{C\alpha\lambda^n}(f^n(x))$$

$$Df^{-n}(x)(\mathcal{C}^E_\alpha(x)) \subset \mathcal{C}^E_{C\alpha\lambda^n}(f^{-n}(x))$$

Deciding whether a given invariant set has a dominated splitting may seem to be tricky, because it may not be clear how to find the subspaces in order to verify the required properties. An alternate way, that can be checked with limited accuracy and that is clearly robust under perturbation is the cone criterion shown below. The interested reader could consult the complete proof in [CP15] for instance.

Proposition 2.1.2 (The Alexeev cone criterion) *Let $f : M \to M$ be a diffeomorphism on a closed manifold M and Λ be any f-invariant set. Suppose that there exist a cone-field $\mathcal{C}^F_\alpha(x)$ in Λ and $\lambda \in (0,1)$ such that:*

$$Df(x)(\mathcal{C}^F_\alpha(x)) \subset \mathcal{C}^F_{\alpha\lambda}(f(x))$$

Then Λ has dominated splitting.

Proof.[Sketch of the proof] For each $x \in \Lambda$ we define:

$$E_x = \bigcap_{n\geq 0} Df^{-n}(f^n(x))(\mathcal{C}^E_\alpha(f^n(x)))$$

and

$$F_x = \bigcap_{n\geq 0} Df^{n}(f^{-n}(x))(\mathcal{C}^F_\alpha(f^{-n}(x)))$$

Here E_x and F_x are Df-invariant and $T_xM = E_x \oplus F_x$. For the domination, given $u_E \in E_x$ and $u_F \in F_x$ two unitary vectors we have there exists a uniform $m \geq 1$ such that $Df^m(x)(u_E + u_F)$ belongs to a small cone around $F_{f^m(x)}$. This implies that $\| Df^m(x)(v_E) \| \leq \frac{1}{2} \| Df^m(x)(v_F) \|$. □

Let us list some useful elementary properties of dominated splittings and the respective proofs (from [BDV05], Appendix B)

a) **Uniqueness:** The dominated splitting is unique if one fixes the dimensions of the subbundles.

Proof. Assume that $E \oplus_< F$ and $G \oplus_< H$ are two dominated splitting over Λ such that $\dim(E) = \dim(G)$. We will show that $E \subset G$ and then $E = G$ and $F = H$.

So, assume there exists $x \in \Lambda$ such that $E(x) \not\subseteq G(x)$ and consider some unit vector $u \in E(x) \backslash G(x)$. Write $u = u_G + u_H$, with $u_G \in G(x)$ and $0 \neq u_H \in H(x)$. Then the positive iterates of u grow at the same rate as those of u_H. Write also $u_H = v_E + v_F$, with $v_E \in E(x)$ and $v_F \in F(x)$.

If $v_F \neq 0$ then the positive iterates of v_F would grow at the same rate as those of u_H, that is, at the same rate as the iterates of $u \in E(x)$, which would contradict the domination $E \oplus_< F$.

Therefore, $v_F = 0$, then $u_H \in E(x) \cap H(x)$. As we are assuming that $E(x) \neq G(x)$ then there is some unit vector $w \in G(x) \backslash E(x)$ which we write $w = w_E + w_F$, with $w_F \neq 0$.

Then the positive iterates of $w \in G(x)$ grow at the same rate as those of w_F, and so the positive iterates of $w \in G(x)$ grow exponentially faster than the iterates of $u_H \in E(x)$. Since u_H is also in $H(x)$, this contradicts the domination $G \oplus_< H$, and completes the proof. □

b) **Continuity:** The splitting $E_1 \oplus_< \ldots \oplus_< E_n$ varies continuously with $x \in \Lambda$.

Proof. Let $E \oplus_< F$ be an l-dominated splitting over Λ. Let $(x_n)_{n \in \mathbb{N}} \subset \Lambda$ be a sequence converging to some point $x \in M$. Without loss of generality we can assume that the spaces $E(x_n)$ and $F(x_n)$ converge to subespaces $\tilde{E}(x)$ and $\tilde{F}(x)$, and the dimension of the spaces are equal, i.e. $\dim(E(x_n)) = \dim(\tilde{E}(x))$ and $\dim(F(x_n)) = \dim(\tilde{F}(x))$. We will show that $\tilde{E}(x) = E(x)$ and $\tilde{F}(x) = F(x)$.

For any $k \in \mathbb{N}$, for any unit vectors $u \in \tilde{E}(x)$ and $v \in \tilde{F}(x)$ we have:

$$\frac{\| Df^{kl}(x)(u) \|}{\| Df^{kl}(x)(v) \|} = \lim_n \frac{\| Df^{kl}(x_n)(u_n) \|}{\| Df^{kl}(x_n)(v_n) \|} \leq \left(\frac{1}{2}\right)^k$$

This characterizes the subespace $\tilde{E}(x)$ uniquely (once its dimension is fixed): the iterates of any unit vector $u \in \tilde{E}(x)$ grow slower than those of any unit vector $w \notin \tilde{E}(x)$. This proved that effectively $\tilde{E}(x) = E(x)$.

Analogously

$$\frac{\| Df^{-kl}(x)(v) \|}{\| Df^{-kl}(x)(u) \|} = \lim_n \frac{\| Df^{-kl}(x_n)(v_n) \|}{\| Df^{-kl}(x_n)(u_n) \|} \leq \left(\frac{1}{2}\right)^k$$

This characterizes the subespace $\tilde{F}(x)$ uniquely and then the splitting dominated varies continuously with $x \in \Lambda$. □

c) **Transversality:** The angles between any two subbundles of a dominated splitting are uniformly bounded from zero.

Proof. Suppose that $T_\Lambda M = E \oplus_< F$. We will show that the angle between E and F is uniformly bounded from zero.

For doing that it is enough to prove that there exists $\alpha > 0$ such that $\| v_E - v_F \| \geq \alpha > 0$ for every pair of unit vectors $v_E \in E(x)$ and $v_F \in F(x)$ independent of $x \in \Lambda$.

Suppose that there exists two sequences $(u_n)_{n\in\mathbb{N}} \subset E(x_n)$ and $(v_n)_{n\in\mathbb{N}} \subset F(x_n)$ of unit vectors such that $u_n - v_n \to 0$. As the derivative of f is bounded, for every $l \in \mathbb{N}$ there exists $u_{n_l} \in E(x_{n_l})$ and $v_{n_l} \in F(x_{n_l})$ such that

$$\frac{\| Df^l(x_{n_l})(u_{n_l} - v_{n_l}) \|}{\| Df^l(x_{n_l})(v_{n_l}) \|} < \frac{1}{2}$$

This implies that for every $l \in \mathbb{N}$ there exists $u_{n_l} \in E(x_{n_l})$ and $v_{n_l} \in F(x_{n_l})$ unitaries such that

$$\frac{1}{2} < \frac{\| Df^l(x_{n_l})(u_{n_l}) \|}{\| Df^l(x_{n_l})(v_{n_l}) \|} < 2$$

which contradicts the domination. □

d) **Extension to the closure:** The splitting $E_1 \oplus_< \ldots \oplus_< E_n$ extends to a dominated splitting over the closure $\overline{\Lambda}$ of Λ.

Proof. The splitting on Λ extends by continuity to the closure of Λ using the argument showed in the part $b)$ □

e) **Persistence:** Every dominated splitting persists under C^1-perturbations.

Proof. Consider a l-dominated splitting $E \oplus_< F$ on Λ. We can extend it to a continuous splitting $TM = E \oplus F$ in a neighborhood U of Λ not necessarily invariant. We consider the cone fields $\mathcal{C}_\alpha^F$ on U defined by

$$\mathcal{C}_\alpha^F(x) = \{v = v_E + v_F \in T_xM : \| v_F \| \geq \alpha \| v_E \|\}, \, x \in U$$

The dominate splitting $T_\Lambda M = E \oplus_< F$ implies that, for any $x \in \Lambda$,

$$Df^l(\mathcal{C}_1(x)) \subset \mathcal{C}_2(f^l(x))$$

Then, for any $\epsilon \in (0,1)$ there is a neighborhood $V \subset U$ of Λ and a C^1-neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ and any $x \in V$ we have:

$$Dg^l(\mathcal{C}_1(x)) \subset \mathcal{C}_{2-\epsilon}(g^l(x))$$

This implies that the maximal invariant set $\bigcap_{n\in\mathbb{Z}} g^n(V)$ of g in V has a dominated splitting with the same dimensions of the initial one, and with almost the same strength. □

2.2 Hyperbolic and partially hyperbolic diffeomorphisms

A compact invariant set $K \subset M$ of a diffeomorphism $f : M \to M$ is *hyperbolic* if the tangent bundle over K splits into two subbundles $T_KM = E^s \oplus E^u$ such that:

(a) E^s and E^u are Df-invariant, i.e. $Df(x)(E^s_x) = E^s_{f(x)}$ and $Df(x)(E^u_x) = E^u_{f(x)}$

(b) There exists $C > 0$ and $\lambda \in (0,1)$ such that for all $v_s \in E^s(x)$, $v_u \in E^u(x)$ and $n \in \mathbb{N}$ we have:

$$\| Df^n(x)(v_s) \| \leq C\lambda^n \| v_s \| \quad \text{and} \quad \| Df^{-n}(x)(v_u) \| \leq C\lambda^n \| v_u \|$$

In this case we say that $Df(x)$ is contracting on $E^s(x)$ and $Df(x)$ is expanding on $E^u(x)$

If $K = M$ we say that f is a hyperbolic diffeomorphism or Anosov diffeomorphism.

We say that f is *partially hyperbolic* on an invariant set $\Lambda \subset M$ if the tangent bundle splits into three nontrivial invariant subbundles E^s, E^c and E^u and $N \in \mathbb{N}$ such that for every $x \in \Lambda$:

(a) $Df^N(x)$ is contracting on $E^s(x)$ and $Df^N(x)$ is expanding on $E^u(x)$.

(b) The splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ is dominated.

If f is partially hyperbolic in Λ we say that Λ is a partially hyperbolic set.

2.2.1 Hyperbolic points

Let p a periodic point of $f : M \to M$ and we denote by $\pi(p)$ the period of p. We say that p is a hyperbolic periodic point of f if p is a periodic point and the derivate

$$D_p f^{\pi(p)} : T_pM \to T_pM$$

has no eigenvalues of modulus 1. We denote by $\mathrm{Per}(f)$ the set of periodic point and also $\mathrm{Per}_H(f)$ the set of hyperbolic periodic point. Is clear that these sets are f-invariant.
If $p \in \mathrm{Per}_H(f)$ we have that there exist subespaces $E^s(p)$, $E^u(p)$ in T_pM such that $T_pM = E^s(p) \oplus E^u(p)$ which are Df-invariant, i.e. $D_pf(E^s(p)) = E^s(f(p))$ and $D_pf(E^u(p)) = E^u(f(p))$. Here $E^s(p)$ is the eigenspace associated to the eigenvalues of modulus smaller than 1 of $D_pf^{\pi(p)}$ and $E^u(p)$ is the eigenspace associated to the eigenvalues of modulus bigger than 1.
We define the unstable index $u(p)$ of a hyperbolic periodic point p as $\dim(E^u(p))$.
The next theorem show that hyperbolic periodic points remain hyperbolic under small C^1-perturbations.

Theorem 2.2.1 *Let $f : M \to M$ be a C^1-diffeomorphism and p a periodic hyperbolic point of f. Then there exist $\mathcal{U}(f)$ a C^1 neighborhood of f and U_p a neighborhood of p such that for every $g \in \mathcal{U}(f)$ there exists p_g a periodic hyperbolic point of g[1] inside U_p which has the same period of p. Moreover, p_g varies continuously in g.*

[1] p_g is called the analytic continuation of p

2.3 Examples of diffeomorphisms with dominated splitting

Here we will present some examples of diffeomorphism with dominated splitting.

1. Hyperbolic diffeomorphism: Let $f : M \to M$ be a diffeomorphism and $\Lambda \subset M$ a hyperbolic set, then the splitting $T_\Lambda M = E^s \oplus E^u$ is dominated. In particular all Anosov diffeomorphism have a dominated splitting.
2. Hyperbolic periodic point: Let $f : M \to M$ be a diffeomorphism and $p \in \mathrm{Per}_H(f)$ then the splitting $T_{\mathcal{O}(p)}M = E^s \oplus E^u$ is dominated.
3. Mañé derived from Anosov [Mañ78]: We start taking an linear Anosov diffeomorphism $f_0 : \mathbb{T}^3 \to \mathbb{T}^3$ with one expanding and two contracting directions and a fixed point p of f_0. Deforming f_0 by isotopy in a neighborhood $V = B(p, \delta)$ of p we have that there exists a C^1-open set $\mathcal{U}$ such that satisfies the following:

   (A) f has a expanding foliation $\mathcal{F}^{uu}$ and a center foliation $\mathcal{F}^c$. These foliations are tangent to the subbundles E^{uu} and E^c and $TM = E^c \oplus_< E^{uu}$, where $\dim(E^{uu}) = 1$ and $\dim(E^c) = 2$

   (B) f has three hyperbolic fixed points inside V, contained in a same central leaf: one fixed point with unstable index 2 and two fixed points with unstable index 1 such that at least one has complex contracting eigenvalues. We can do it passing the periodic point through a Hopf bifurcation.

   (C) There exists $\sigma > 1$ such that $|\det(Df^{-1}|_{E^c})| \geq \sigma$.

   Bonatti - Viana [BV00] proved that for every $f \in \mathcal{U}$ as before the foliation $\mathcal{F}^{uu}$ is minimal, the largest Lyapunov exponent $\lambda_+^c(x)$ of f along the bundle E^c is negative for Lebesgue almost every point in any segment contained in a leaf of $\mathcal{F}^{uu}$. Also, they proved that these diffeomorphism are stably ergodic.
4. Bonatti-Viana example in $\mathbb{T}^4$ [BV00]: As before, we start with a linear Anosov diffeomorphism $f_0 : \mathbb{T}^4 \to \mathbb{T}^4$ induced by a linear map of $\mathbb{R}^4$ with eigenvalues
$$0 < \lambda_1 \leq \lambda_2 < \frac{1}{3} < 1 < 3 < \lambda_3 \leq \lambda_4$$
   and dominated splitting $T\mathbb{T}^4 = (E^{ss} \oplus E^s) \oplus_< (E^u \oplus E^{uu})$. Up to rem-

placing it by some iterate, we can suppose that f_0 has at least two fixed points p_1 and p_2.

Let $V = B(p_1, \delta) \cup B(p_2, \delta)$ be a union of balls centered at p_1, p_2 and radius $\delta > 0$ sufficiently small. Deforming the Anosov diffeomorphism inside V passing through a pitchfork bifurcation along $E^{ss} \oplus E^{s}$ and then another deformation to obtain one fixed point with complex contracting eigenvalues.

We obtain a new diffeomorphism with dominated splitting $T\mathbb{T}^4 = E^{cs} \oplus_{<} (E^{u} \oplus E^{uu})$, where $\dim(E^{cs}) = 2$.

After that, we do the same for p_2, but in the unstable direction. Finally we obtain a C^1-open set $\mathcal{U}$ of diffeomorphism with dominated splitting $T\mathbb{T}^4 = E^{cs} \oplus_{<} E^{cu}$, where $\dim(E^{cs}) = \dim(E^{cu}) = 2$ without invariant hyperbolic subbundles.

Bonatti-Viana [BV00] proved that each f is a robustly transitive diffeomorphism and Tahzibi [Tah04] proved the stable ergodicity.

2.4 Lyapunov exponents

Let $f : M \to M$ be a C^1-diffeomorphism of a compact Riemannian manifold of dimension d. Given $v \in T_xM$, the Lyapunov exponent of v is:

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \| Df^n(x)(v) \| .$$

let $E_\lambda(x)$ be the subspace of T_xM consisting of all v such that the Lyapunov exponent de v is λ. We have the well-known Oseledets' Theorem.

Theorem 2.4.1 (Oseledets [Ose68]) *There is an f-invariant Borel set $\mathcal{D}$ of total probability (in the sense that $\mu(\mathcal{D}) = 1$ for all f-invariant probability measures μ), and for each $\epsilon > 0$ exists a Borel function $C_\epsilon : \mathcal{D} \to (1, +\infty)$ such that $\forall x \in \mathcal{D}$, $v \in T_xM$ y $n \in \mathbb{Z}$:*

1. *There exist a splitting (called the Oseledets' splitting) of the tangent bundle*

   $$T_xM = E_1(x) \oplus \ldots \oplus E_{k(x)}$$

   and numbers $\lambda_1(x) < \ldots < \lambda_{k(x)}(x)$ such that for each vector in the subspace $E_i(x)$ its associated Lyapunov exponent is $\lambda_i(x)$.

2. $\frac{1}{C_\epsilon(x)} e^{(\lambda-\epsilon)n} \parallel v \parallel \leq \parallel Df^n(x)v \parallel \leq C_\epsilon(x) e^{(\lambda+\epsilon)n} \parallel v \parallel$, $\forall v \in E_\lambda(x)$.
3. $C_\epsilon(f(x)) \leq e^\epsilon C_\epsilon(x)$.
4. $\angle(E_\lambda(x), E_\gamma(x)) \geq \frac{1}{C_\epsilon(x)}$, $\forall \lambda \neq \gamma$.

The set $\mathcal{D}$ is called the set of regular points. We have that $Df(x)E_\lambda(x) = E_\lambda(f(x))$ and if an f-invariant measure μ is ergodic, then the Lyapunov exponents and $\dim E_\lambda(x)$ are constant μ-almost everywhere.
For all $x \in \mathcal{D}$, we have

$$T_x M = \bigoplus_{\lambda<0} E_\lambda(x) \oplus E^0(x) \bigoplus_{\lambda>0} E_\lambda(x)$$

where $E^0(x)$ is the subspace generated by the vectors having zero Lyapunov exponents.
Let $\text{Diff}^r_m(M)$ be the set of C^r-conservative diffeomorphisms (i.e. preserving a smooth volume form m) endowed with the C^r topology. $\text{Diff}^r_m(M)$ is a Baire space for any integer $r \geq 0$ (see [PdM78]). In a Baire space, a set is *residual* if it contains a countable intersection of dense open sets. We establish a convention: the phrases "generically f satisfy..." and "every generic diffeomorphism f satisfies..." should be read as "there exists a residual subset $\mathcal{R} \subset \text{Diff}^1_m(M)$ such that every $f \in \mathcal{R}$ satisfies..."
Let $\lambda_1(x) \leq \lambda_2(x) \leq \ldots \leq \lambda_d(x)$ be the Lyapunov exponents with multiplicities, then if $f \in \text{Diff}^1_m(M)$ we have $\lambda_1(x) + \lambda_2(x) + \ldots + \lambda_d(x) = 0$.

For fixed $\epsilon > 0$ and given $L > 0$, we define the *Pesin blocks* by

$$\mathcal{D}_{\epsilon,L} = \{x \in \mathcal{D} : C_\epsilon(x) \leq L\}$$

Note that Pesin blocks are not necessarily invariant, although $f(\mathcal{D}_{\epsilon,L}) \subseteq \mathcal{D}_{\epsilon,e^\epsilon L}$. Also, for each $\epsilon > 0$, we have

$$\mathcal{D} = \bigcup_{L=1}^{\infty} \mathcal{D}_{\epsilon,L}$$

Since, the Lebesgue measure is regular, without loss of generality we can asume that the Pesin blocks are compact.

A diffeomorphism $f \in \text{Diff}^1_m(M)$ is *non-uniformly hyperbolic* if all Lyapunov exponents are non-zero m-almost everywhere, that is if for m-almost every x, and every unit vector $v \in T_xM$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| \neq 0$$

A diffeomorphism $f \in \text{Diff}^1_m(M)$ is *stably non-uniformly hyperbolic* if there exists a neighborhood $\mathcal{U}$ of f in $\text{Diff}^1_m(M)$ such that all diffeomorphisms g in $\mathcal{U} \cap \text{Diff}^2_m(M)$ are non-uniformly hyperbolic.

2.5 Hausdorff Topology

Given (M, d) a compact metric space, given two non-empty compact sets $A, B \subset M$ we define the Hausdorff distance:

$$d_H(A, B) = \inf\{\epsilon \geq 0 : A \subset B_\epsilon(B), \quad B \subset B_\epsilon(A)\}$$

where $B_\epsilon(A) = \bigcup_{a\in A} B_\epsilon(a)$.
By convention, the Hausdorff distance from the empty set to any non-empty set is equal to diameter of M.
The set $\mathcal{K} = \{K \subset M : K \text{ is compact and non-empty}\}$ is a compact metric space with the Hausdorff distance (see [KH95]).

2.6 Dominated splitting and periodic points

The main result in this section says that if a diffeomorphism f has a global dominated splitting $TM = E \oplus_< F$ and every periodic point p of f has unstable index equal to $\dim(F)$ and this happens for every g in a neighborhood of f then f is an Anosov diffeomorphism. To prove the result above we use two results borrowed from [BDPR00] and originally due to Ricardo Mañé [Mañ82]. This result is true in the volume-preserving case.

The next proposition is extracted from [BDPR00] which is a reformulation of [[Mañ82], Proposition II.1].

Proposition 2.6.1 *Let $f \in \mathrm{Diff}^1(M)$ and let Λ be a compact f-invariant set having a dominated splitting $T_\Lambda M = E \oplus_< F$. If there exists a neighborhood U of Λ and $\mathcal{U}(f) \subset \mathrm{Diff}^1(M)$ a neighborhood of f such that every $g \in \mathcal{U}(f) \cap \mathrm{Diff}^1(M)$ has not hyperbolic points of unstable index different of $\dim(F)$. Then there exists $\mathcal{V} \subset \mathrm{Diff}^1(M)$ a neighborhood of f and constants $K > 0$, $m \in \mathbb{N}$ and $\lambda \in (0,1)$ such that for every periodic point x of g whose orbit is contained in U we have:*

(a) If x has period $n \geq m$ then

$$\prod_{i=0}^{k-1} \| Dg^m(g^{mi}(x))|_{E_g(g^{mi}(x))} \| \leq K\lambda^k$$

where k is the entire part of $\frac{n}{m}$.

(b) Moreover,

$$\limsup_{r\to+\infty} \frac{1}{r} \sum_{i=0}^{r-1} \log \| Dg^m(g^{mi}(x))|_{E_g(g^{mi}(x))} \| < 0$$

The next theorem is the clasical Mañé's Ergodic Closing Lemma which we enunciate here by completeness. We remark that it is valid in the volume-preserving case.

Theorem 2.6.2 ([Mañ82], Theorem A) *Given $f \in \mathrm{Diff}^1(M)$ there exists a f-invariant set $\Sigma(f)$, named set of well closable points of f, such that:*

(a) The set $\Sigma(f)$ has total measure.

(b) For every $x \in \Sigma(f)$ and $\epsilon > 0$ there is a diffeomorphism g, which is ϵ-close to f with the C^1-topology, such that x is a periodic point for g and the distance $d(f^i(x), g^i(x)) < \epsilon$ for all $i \in [0, \pi(x,g)]$, where $\pi(x,g)$ is the period of x respect to g.

Theorem 2.6.3 *Let $f \in \mathrm{Diff}^1_m(M)$ and let Λ be a compact f-invariant set having a dominated splitting $T_\Lambda M = E \oplus_< F$. If there exists a neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^1(M)$ of f such that every $g \in \mathcal{U}(f) \cap \mathrm{Diff}^1_m(M)$ has not hyperbolic points of unstable index different of $\dim(F)$. Then Λ is a hyperbolic set.*
In particular, if $\Lambda = M$ then f is an Anosov diffeomorphims.

Proof. By compactness of Λ, to get the hyperbolicity it is enough to see that

$$\liminf_{n\to+\infty} \| Df^n(x)|_{E(x)} \| = 0 \tag{2.2}$$

$$\liminf_{n\to+\infty} \| Df^{-n}(x)|_{F(x)} \| = 0 \tag{2.3}$$

for all $x \in \Lambda$. We will prove 2.2 because 2.3 is similar if we apply the same methods to f^{-1} instead of f.
Suppose by contradiction that 2.2 does not hold for every $x \in \Lambda$, then we can find $x_0 \in \Lambda$, $\kappa > 0$ and $n_0 \in \mathbb{N}$ such that

$$\| Df^n(x_0)|_{E(x_0)} \| > \kappa > 0$$

for every $n \geq n_0$.
We take m as in the proposition 2.6.1 and we consider a sequence of probabilities measures $\{\mu_n\}$ defined by:

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^{mi}(x_0)}$$

where δ_z is the Dirac measure at the point z. Taking a subsequence of $\{\mu_n\}$, we can assume that $\{\mu_n\}$ is converges to a probability measure μ with the weak topology, this is

$$\int \varphi\, d\mu = \lim_{n\to+\infty} \int \varphi\, d\mu_n = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^{mi}(x_0))$$

for every $\varphi : M \to \mathbb{R}$ continuous. Here the function

$$x \longmapsto \log \| Df^m(x)|_{E(x)} \|$$

is continuous, then

$$\int \log \| Df^m(x)|_{E(x)} \| \; d\mu = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \| Df^m(f^{mi}(x_0))|_{E(f^{mi}(x_0))} \|$$

and by the election of x_0

$$\int \log \| Df^m(x)|_{E(x)} \| \; d\mu \geq \lim_{n\to+\infty} \frac{1}{n} \log \| Df^{nm}(x_0))|_{E(x_0)} \| \geq \lim_{n\to+\infty} \frac{\log(\kappa)}{n} = 0$$

then

$$\int \log \parallel Df^m(x)|_{E(x)} \parallel \, d\mu \geq 0 \tag{2.4}$$

On the other hand, by Birkhoff's Theorem

$$\int \log \parallel Df^m(x)|_{E(x)} \parallel \, d\mu = \int \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \parallel Df^m(f^{mi}(x))|_{E(f^{mi}(x))} \parallel \, d\mu \tag{2.5}$$

By Mañé's Ergodic Closing Lemma 2.6.2 we have $\Lambda \cap \Sigma(f)$ is an f-invariant total probability subset of Λ.
By the equations 2.4 and 2.5 we get a point $p \in \Lambda \cap \Sigma(f)$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \parallel Df^m(f^{mi}(p))|_{E(f^{mi}(p))} \parallel \geq 0$$

By item (b) of Proposition 2.6.1, the point p is not periodic for f. By Mañé's Ergodic closing Lemma given $\epsilon > 0$ there exists $g \in \text{Diff}^1_m(M)$ arbitrarily C^1-close to f such that p is a periodic point of g with period $\pi_g(p)$ and the distance $d(f^i(p), g^i(p)) < \epsilon$, for every $i = 0, 1, \ldots, \pi_g(p)$.
Observe that since p is not periodic for f we have if $g_n \to g$ then $\pi_{g_n}(p)$ goes to infinity.
Since the fibers $E_g(y)$ varies continuosly with (y, g), then the function:

$$(y, g) \longmapsto \log \parallel Dg^m(y)|_{E_g(y)} \parallel$$

is continuous.
By item (b) of Proposition 2.6.1, let $\lambda_0 < 1$ and $n_0 \in \mathbb{N}$ such that $\lambda < \lambda_0 < 1$ and for every $n \leq n_0$ we have:

$$\frac{1}{n}\sum_{i=0}^{n-1} \log \parallel Df^m(f^{mi}(p)) \parallel \leq \frac{1}{2}\log(\lambda_0)$$

We can also assume that $K\lambda^n < \lambda_0^n$, for all $n \geq n_0$. So if g is close enough to f twe have

$$|\log \parallel Dg^m(g^i(p)) \parallel - \log \parallel Df^m(f^i(p)) \parallel | < \frac{1}{2}|\log(\lambda_0)|$$

for every $i \in [0, \pi_g(p)]$.

Moreover, we can assume that the entire part k_g of $\frac{\pi_g(p)}{m}$ is greater than n_0. Thus,

$$\frac{1}{k_g}\sum_{i=0}^{n-1}\log \parallel Dg^m(g^i(p)) \parallel \geq \frac{1}{2}\log(\lambda_0) \geq \frac{1}{2}\log(\lambda_0^k) > \frac{1}{2}\log(K\lambda^k)$$

contradicting item (a) of Proposition 2.6.1. □

2.7 The Pesin Homoclinic class

For $x \in M$, we define the Pesin stable set of x as:

$$W^-(x) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0\}$$

and analogously the Pesin unstable set:

$$W^+(x) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0\}$$

Stable and unstable Pesin sets of points in the set of regular points $\mathcal{D}$ are immersed manifolds (see [Pes77]).
Given a hyperbolic periodic point $p \in M$, we define the *stable Pesin homoclinic class* [1] of p by

$$\mathrm{Phc}^-(p) = \{x \in M : W^-(x) \pitchfork W^+(o(p)) \neq \emptyset\}$$

where $W^u(o(p))$ is the union of the unstable manifolds of $f^k(p)$, for all $k = 0, \ldots, \mathrm{per}(p) - 1$. $\mathrm{Phc}^-(p)$ is invariant and saturated by W^--leaves. Analogously, we define

$$\mathrm{Phc}^+(p) = \{x \in M : W^+(x) \pitchfork W^-(o(p)) \neq \emptyset\}$$

If there exists an expanding foliation W^u, we will denote

$$\mathrm{Phc}^u(p) = \{x \in M : W^u(x) \pitchfork W^-(o(p)) \neq \emptyset\}$$

Analogously we define $\mathrm{Phc}^s(p)$ if a contracting foliation W^s is given. The fo-

[1] This set was defined in [HHTU11] and it is called ergodic homoclinic class.

liation will be clear from the context, if it is not, we will denote these sets by $\mathrm{Phc}^W(p)$, where W is given.
Observe that if there exists an expanding foliation W^u then $\mathrm{Phc}^u(p) \subset \mathrm{Phc}^+(p)$.
A useful tool to work with a transversal intersection between stable and unstable manifolds is the λ-lemma of Palis [Pal69].

Theorem 2.7.1 (λ-Lemma) *Let $f : M \to M$ be a C^1-diffeomorphism and p a fixed hyperbolic point of f. Let D^u a compact disk in $W^+(p)$ and let D be a manifold of equal dimension of $W^+(p)$ such that $D \pitchfork W^-(p) \neq \phi$.*
Then $\forall \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $D_n \subset D$ such that $f^n(D_n)$ and D^u are ϵ C^1-closed.

The importance of Pesin homoclinic classes comes from the next criterion of ergodicity:

Theorem 2.7.2 (Theorem A, [HHTU11]) *Let $f : M \to M$ be a C^2-diffeomorphism over a closed connected Riemannian manifold M, let m be a smooth invariant measure and $p \in \mathrm{Per}_H(f)$. If $m(\mathrm{Phc}^+(p)) > 0$ and $m(\mathrm{Phc}^-(p)) > 0$, then*

1. *$\mathrm{Phc}^+(p) \mathrel{\mathring{=}} \mathrm{Phc}^-(p) \mathrel{\mathring{=}} \mathrm{Phc}(p)$, where $\mathrm{Phc}(p) = \mathrm{Phc}^+(p) \cap \mathrm{Phc}^-(p)$.*
2. *$m|\,\mathrm{Phc}(p)$ is ergodic.*
3. *$\mathrm{Phc}(p) \subset \mathrm{Nuh}(f)$, where $\mathrm{Nuh}(f)$ is the set of x in M such that all Lyapunov exponents of x are different from zero.*

We have an ergodic analogous to Smale's spectral decomposition theorem combinating Pesin's ergodic component theorem [Pes77], the ergodicity criterion statement above and the next theorem:

Theorem 2.7.3 (Katok's closing lemma, [Kat80]) *Let M be a compact Riemannian manifold of finite dimension and let $f : M \to M$ be a C^2-diffeomorphism. Then for every $k = 0, \dots, \dim M$, and for all ϵ, $L > 0$, there exists $r > 0$ such that if:*

1. *x, $f^n(x) \in \mathcal{D}^k_{\epsilon,L}$, for some $n > 0$, where*
   $\mathcal{D}^k_{\epsilon,L} = \mathcal{D}_{\epsilon,L} \cap \{x \in \mathcal{D} / \dim E^u(x) = k\}$
2. *$d(x, f^n(x)) < r$*

Then there exists $p \in \mathrm{Per}_H(f)$ such that $x \in \mathrm{Phc}(p)$.

Theorem 2.7.4 *Let M be a closed connected Riemannian manifold, let $f : M \to M$ be a C^2-diffeomorphism and let m be a smooth measure hyperbolic over an f-invariant set V. Then:*

(a) We have:

$$V \overset{\circ}{=} \bigcup_{n \in \mathbb{N}} \Lambda_n$$

where Λ_n are disjoint measurable invariant sets such that $f|_{\Lambda_n}$ is ergodic.

(b) For each Λ_n, there exists $k_n \in \mathbb{N}$ and measurable sets with positive measure $\Lambda_1^n, \Lambda_2^n, \ldots, \Lambda_{k_n}^n$ which are pairwise disjoints such that $f(\Lambda_j^n) = \Lambda_{j+1}^n$ for every $j = 1, 2, \ldots, k_n - 1$, $f(\Lambda_{k_n}^n) = \Lambda_1^n$ and f^{k_n} is Bernoulli.

(c) There exists a hyperbolic periodic point p_n such that $\Lambda_n = \mathrm{Phc}(p_n)$

Proof. We will show the items (a) and (c). The item (b) is given by the Pesin's ergodic component theorem [Pes77].
Let $\epsilon, L > 0$ be and $k = 0, 1, \ldots, \dim M$ such that $m\left(\mathcal{D}_{\epsilon,L}^k\right) > 0$ and let x be a density point of $\mathcal{D}_{\epsilon,L}^k$. Take $r > 0$ given by the Katok's closing lemma, due to x is a density point of $\mathcal{D}_{\epsilon,L}^k$ we have $m\left(\mathcal{D}_{\epsilon,L}^k \cap B_{r/2}(x)\right) > 0$, then by Poincaré's recurrence theorem there exists $n > 0$ such that $f^n(x) \in \mathcal{D}_{\epsilon,L}^k \cap B_{r/2}(x)$, then by Katok's closing lemma there exists $p \in Per_H(f)$ such that $x \in \Lambda(p)$. In conclusion we have proved that:

$$\mathcal{D}_{\epsilon,L}^k \overset{\circ}{\subset} \Lambda(p), \quad \text{for some} \quad p \in Per_H(f)$$

Fixing $\epsilon > 0$ we have:

$$M \overset{\circ}{=} \bigcup_{L,k} \mathcal{D}_{\epsilon,L}^k, \quad con \quad L \in \mathbb{N}, \quad k = 0, \ldots, \dim M$$

and then there exists a sequence of hyperbolic periodic points such that

$$M \overset{\circ}{=} \Lambda_1 \cup \ldots \Lambda_n \cup \ldots, \quad \text{with} \quad \Lambda_i = \Lambda(p_i), \quad p_i \in Per_H(f)$$

Here $\mathcal{D}_{\epsilon,L}^k \subset \Lambda(p_i)$ for some $p_i \in Per_H(f)$ and $m\left(\mathcal{D}_{\epsilon,L}^k\right) > 0$ then $m\left(\Lambda(p_i)\right) > 0$. By the criterion of ergodicity 2.7.2 $f|_{\Lambda_i}$ is ergodic and clearly the sets Λ_i are measurables and f-invariant. Moreover these sets are disjoint, because if p and q are hyperbolic periodic points such that $\Lambda(p) \cap \Lambda(q) \neq \phi$

then there exists a point z homoclinically related with p and q, using the λ-Lemma we have $\Lambda(p) = \Lambda(q)$.

$\square$

Corollary 2.7.5 *In the hypothesis of 2.7.2 if* $m(\text{Phc}(p)) = 1$ *then* f *is Bernoulli.*

Proof. If $m(\text{Phc}(p)) = 1$ we have:

$$M \stackrel{\circ}{=} \Lambda_1^1 \cup \Lambda_2^1 \cup \ldots \cup \Lambda_k^1$$

where the sets Λ_j^1 are measurable and pairwise disjoints. As $f(\Lambda_j^1) = \Lambda_{j+1}^1$ for every $j = 1, 2, \ldots, k-1$ and $f(\Lambda_k^1) = \Lambda_1^1$ we have that $m(\Lambda_j^1) = \frac{1}{k} > 0$, for all $j = 1, 2, \ldots, k$.
Here f^k is ergodic and each Λ_j^1 is f^k-invariant then $k = 1$. This implies that f is Bernoulli. $\square$

Proposition 2.7.6 $\text{Phc}^+(p)$ *satisfy the following:*

a) $\text{Phc}^+(p)$ *is an u-saturated open set.*
b) If $f \in \text{Diff}_m^1(M)$ *is ergodic then* $m(\text{Phc}^+(p)) = 1$*.*

Proof. By transversality, if $W^+(x) \pitchfork W^-(p) \neq \phi$ then exists an open neighborhood U of x in M such that for every $y \in U$ we have $W^+(y) \pitchfork W^-(p) \neq \phi$, so $\text{Phc}^u(p)$ is an open set.
Here f is ergodic, $m(\text{Phc}^+(p)) > 0$ (because it is a nonempty open set) and $\text{Phc}^+(p)$ is f-invariant so $m(\text{Phc}^+(p)) = 1$. $\square$
We denote by:

$$\Lambda(f) = M \backslash \text{Phc}^+(p) \tag{2.6}$$

So we have the next corollary:

Corollary 2.7.7 *The set* $\Lambda(f)$ *is a compact,* f*-invariant, u-saturated subset of* M*. Also, if* $f \in \text{Diff}_m^1(M)$ *is ergodic then* $m(\Lambda(f)) = 0$

I will present two theorems for volume-preserving maps.
The first is the volume-preserving version of the Kupka-Smale Theorem, see [Rob70]:

Theorem 2.7.8 *Assume* $\dim M \geq 3$*,* $r \in \mathbb{Z}^+$*. Then generically in* $\mathrm{Diff}^r_m(M)$*, every periodic orbit is hyperbolic, and for every pair of periodic points* p *and* q*, the manifolds* $W^+(p)$ *and* $W^-(q)$ *are transverse.*

The next is a connecting property due to Arnaud [Arn01] and Bonatti-Crovisier [BC04].

Theorem 2.7.9 *Assume* $\dim M \geq 3$*. Then generically in* $\mathrm{Diff}^1_m(M)$*, if* p *and* q *are periodic points with* $\dim W^+(p) \geq \dim W^+(q)$*, then* $W^+(O(p)) \cap W^-(O(q))$ *is dense in* M*.*

Remark 2.7.1 *[Remark 4.4, [HHTU11]] If* $W^+(p) \pitchfork W^-(q) \neq \phi$ *then* $\mathrm{Phc}^+(p) \subset \mathrm{Phc}^+(q)$ *and* $\mathrm{Phc}^-(q) \subset \mathrm{Phc}^-(q)$

Remark 2.7.2 *Generically in* $\mathrm{Diff}^1_m(M)$ *if* p *and* q *have the same unstable index, then by theorem 2.7.9 the manifolds* $W^+(o(p))$ *and* $W^-(o(q))$ *have nonempty intersection which is transverse by theorem 2.7.8 and by 2.7.1 we have* $\mathrm{Phc}^+(p) = \mathrm{Phc}^+(q)$*. This implies that* $\Lambda(g)$ *is not depending of the hyperbolic periodic point.*

2.8 Blenders

In this section, we will present the concept of Blenders given in [HHTU10]. We warn the reader that there are other definitions of blenders (see for instance [BDV05], chapter 6). Also, in [BDV05] there is a discussion on different ways of defining these objects.
A diffeomorphism $f : M \to M$ has a *heterodimensional cycle* associated with two hyperbolic periodic points p and q of f if their unstable indices are different, the stable manifold $W^-(p)$ of p meets the unstable manifold $W^+(q)$ of q, and the unstable manifold $W^+(p)$ of p meets the stable manifold $W^-(q)$ of q.
We say that p and q are a *co-index one heterodimensional cycle* when the indices of p and q differ in one.
Let p be a partially hyperbolic periodic point for f such that the derivate Df is expanding on E^c and $\dim E^c = 1$. A small open set $Bl^{cu}(p)$, near p but not necessarily containing p, is a *cu-blender near* p if:

1. Every $(u+1)$-strip well placed in $Bl^{cu}(p)$ transversely intersects $W^-(p)$.

2. This property is C^1 -robust. Moreover, the open set associated with the periodic point contains a uniformly sized ball.

A $(u+1)$-strip is any $(u+1)$-disc containing a u-disc D^{uu}, so that D^{uu} is centred at a point in $Bl^{cu}(p)$. The radius of D^{uu} is much bigger than the radius of $Bl^{cu}(p)$ and D^{uu} is almost tangent to E^u, i.e. the vectors tangent to D^{uu} are C^1-close to E^u.
A $(u+1)$-strip is well placed in $Bl^{cu}(p)$ if it is almost tangent to $E^c \oplus E^u$.

For *cs-blenders* we can define similarly considering a partially hyperbolic point such that E^c is one dimensional and Df is contracting on E^c.

Given p' a partially hyperbolic periodic point of f such that Df is expanding on E^c, with $\dim E^c = 1$. A small open set B is called *cu-blender associated with* p' if $B = Bl^{cu}(p)$, where p is a partially hyperbolic periodic point homoclinically related to p' and $Bl^{cu}(p)$ is a cu-blender near p.

The next theorem allows obtaining conservative diffeomorphisms admitting blenders near conservative diffeomorphisms with a pair of hyperbolic periodic points with co-index one.

Theorem 2.8.1 (Theorem 1.1 - [HHTU10]) *Let $f \in \mathrm{Diff}^r_m(M)$ be such that f has two hyperbolic periodic points q and p of unstable indices $(u+1)$ and u respectly. Then there are C^r diffeomorphisms arbitrarily C^1-close to f which preserve m and admits a cu-blender associated with the analytic continuation of q. Moreover p and q form a co-index one heterodimensional cycle.*

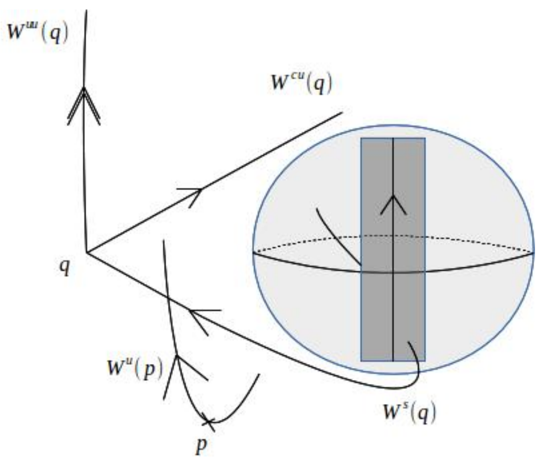

Figure 2.1: *cu*-blender asociated to q

The next remark says that $W^+(p)$ has one more dimension than it should. This property is C^1-robust.

Remark 2.8.1 *In the context of the previous theorem we have*

$$W^+(q) \subset \overline{W^+(p)}$$

Proof. Given $x \in W^+(q)$ and let U_x be a neighborhood of x. Then by λ-lemma, for n big enough $f^n(U_x)$ intersects the cu-blender and then it contains a C^1-open set of $(u+1)$-strip well placed. As p and q form a co-index one heterodimensional cycle we have $W^+(p) \cap f^n(U_x) \neq \phi$ and by the invariance of the unstable manifold $W^+(p) \cap U_x \neq \phi$. This proved $W^+(q) \subset \overline{W^+(p)}$. $\square$

Chapter 3

Stably Bernoulli diffeomorphisms

This chapter is the central objective of the thesis. We will present some result to be able to show the main theorems 1.1.1 and 1.1.2.

3.1 Proof of Theorem 1.1.3 and the minimality criterion 1.1.4

Before to do the proof of the theorem 1.1.3 and the minimality criterion 1.1.4 let's see the next result about the semi-continuity of the map $f \longmapsto \Lambda(f)$, where $\Lambda(f)$ is the set defined in 2.6.

Lemma 3.1.1 *With the Hausdorff topology, the function $f \longmapsto \Lambda(f)$ is upper-semicontinuous, that is: if $f_n \overset{C^1}{\to} f$ then $\limsup_{n\to\infty} \Lambda(f_n) \subset \Lambda(f)$.*

Proof. Suppose that $\{\Lambda(f_n)\}$ is a sequence nonincreasing of non-empty compact sets, then we have:

$$K = \limsup_{n\to\infty} \Lambda(f_n) = \cap_{n=1}^{\infty} \Lambda(f_n)$$

Here $K \neq \phi$ by the finite intersection property, on the other hand if $x \in K$ then $x \in \Lambda(f_n), \quad \forall n \in \mathbb{N}$. We can take n sufficiently large such that f_n is sufficiently close to f with the C^1-topology, then if $x \in \Lambda(f_n)$ we have $x \notin Phc^+_{f_n}(p_n)$, where p_n is the analytic continuation of p. So from here we

can deduce that $x \notin Phc_f^+(p) \Rightarrow x \in \Lambda(f)$, then $K \subset \Lambda(f)$.
If the sets $\{\Lambda(f_n)\}$ are not nonincreasing, we define

$$A_n = \overline{\cup_{i=n}^{\infty} \Lambda(f_i)}$$

then $\{A_n\}$ is a family nonincreasing of non-empty compact sets and

$$K = \limsup_{n\to\infty} \Lambda(f_n) = \limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$$

If $x \in K$ then $x \in A_n = \overline{\cup_{i=n}^{\infty} \Lambda(f_i)}$, $\forall n \in \mathbb{N}$, therefore we can to find a sequence $\{x_n\}$ such that $x_n \in \Lambda(f_n)$, $\forall n \in \mathbb{N}$ and $x_n \to x$, this imply that $x \in \Lambda_f$. ☐

Corollary 3.1.2 *If $\Lambda(f) = \phi$ then $\exists\, \mathcal{U}(f)$ such that $\Lambda(g) = \phi$, $\forall g \in \mathcal{U}(f)$.*

Proof. Suppose that for every $\varepsilon_n = \frac{1}{n}$ there exists $g_n \in \mathcal{U}_n(f) = B(f, \varepsilon_n)$ such that $\Lambda(g_n) \neq \phi$. Then we have the sequence $g_n \stackrel{C^1}{\to} f$ and then by last lemma $\limsup_{n\to\infty} \Lambda(g_n)$ is a non-empty compact set incluid in $\Lambda(f)$, this is absurd because $\Lambda(f) = \phi$. ☐

Proof.[Proof of minimality Criterion 1.1.4]

Step 1 *The leaf of each point in x not only intersects $W^-(o(p))$, but $W^-(p)$ itself, that is:*

$$\mathrm{Phc}^u(p) = \{x \in M : W^u(x) \pitchfork W^-(p) \neq \phi\} = M$$

Let $x \in \mathrm{Phc}^u(p)$ then there exists $k_0 \in \mathbb{Z}$ such that $W^u(x) \pitchfork W^-(f^{k_0}(p)) \neq \phi$. Consider $l \in \mathbb{N}$ the period of p, then by λ-lemma $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that the unstable manifolds $W^u(f^{ln}(x))$ and $W^u(f^{k_0}(p))$ are $\varepsilon - C^1$ closed. Here the unstable manifold of p is dense, then $W^u(p) \pitchfork W^-(f^k(p)) \neq \phi, \forall k \in \mathbb{Z}$. In particular $W^u(f^{k_0}(p)) \pitchfork W^-(p) \neq \phi$, so taking $\varepsilon > 0$ small enough we have $W^u(f^{ln}(x)) \pitchfork W^-(p) \neq \phi$ and then $W^u(x) \pitchfork W^-(p) \neq \phi$ as desired.

Step 2 *There exists $L > 0$ such that $W^u(x) \pitchfork W_L^-(p) \neq \phi$ for every $x \in M$. Here we call $W_L^-(p)$ the set of points that can be joined to p inside $W^-(p)$ by an arc of length less than L, for each $L > 0$. Indeed, let*

$$\Lambda_n := \{x \in M : W^u(x) \cap W_n^-(p) = \phi\},\ n \in \mathbb{N}$$

Clearly Λ_n is a compact, u-saturated set. Suppose that $\Lambda_n \neq \phi$, $\forall n \in \mathbb{N}$ then $\{\Lambda_n\}$ is a sequence nonincreasing of non-empty compact sets satisfying the finite intersection property and then the intersection is a non-empty compact set.

Moreover, $\Lambda := \bigcap_{n\in\mathbb{N}} \Lambda_n \neq \phi$ is a compact, u-saturated set. For all $x \in \Lambda$ we have $W^u(x) \cap W^-(p) = \phi$, which is absurd, because $W^u(x) \pitchfork W^-(p) \neq \phi$, for all $x \in M$. Therefore there exists $L > 0$ such that $W^u(x) \pitchfork W^-_L(p) \neq \phi$.

Step 3 *For each $\varepsilon > 0$ for each $x \in M$ $W^u(x) \pitchfork W^-_\varepsilon(p) \neq \phi$*
Given $\varepsilon > 0$ and $x \in M$ iterating $W^-_\varepsilon(p)$ kl-times for the past, we have $f^{-kl}(W^-_\varepsilon(p)) \supset W^-_L(p)$. Due to $W^u(f^{-kl}(x)) \pitchfork W^-_L(p) \neq \phi$ then $W^u(x) \pitchfork W^-_\varepsilon(p) \neq \phi$. We have proved that $W^u(x) \pitchfork W^-_\varepsilon(p) \neq \phi$, $\forall \varepsilon > 0$ and $\forall x \in M$.

Step 4 *For every $\varepsilon > 0$ and every $x \in M$, $W^u(x)$ is ε-dense.*
Let $L' > 0$ be such that $W^u_{L'}(p)$ is $\frac{\varepsilon}{2}$-dense and let $\delta > 0$ be such that if $d(x,y) < \delta$ then $d_H(W^u_{L'}(x), W^u_{L'}(y)) < \frac{\varepsilon}{2}$, where d_H is the Hausdorff distance. Now $W^u(x) \pitchfork W^-_\delta(p) \neq \phi$, $\forall x \in M$, then there exists $y \in W^u(x) \cap W^-_\delta(p)$ such that $d_H(W^u_{L'}(y), W^u_{L'}(p)) < \frac{\varepsilon}{2}$, so $W^u(x) \supset W^u_{L'}(p)$ is ε-dense.

Since this holds for all $\varepsilon > 0$, W^u is minimal. □

Now, We will show Theorem 1.1.3

Proof.[Proof of Theorem 1.1.3]

Step 1 *f is stably ergodic.*
As before, for all $g \in \mathcal{U}(f)$ we define:

$$\Lambda(g) := M \setminus \mathrm{Phc}^+(p_g)$$

Here $\Lambda(g)$ is a compact invariant set and by lemma 3.1.1 the map $g \mapsto \Lambda(g)$ varies upper-semicontinuously. By hypothesis, W^u is an f-invariant expanding minimal foliation, thus $\Lambda(f) = \phi$. This implies by the upper-semicontinuity that $\Lambda(g) = \phi$ in a C^1-neighborhood, which we still call $\mathcal{U}(f)$.
Due to $\Lambda(g) = \phi$ then $\mathrm{Phc}^+(p_g) = M$. By hypothesis $m(\mathrm{Phc}^-(p_g)) > 0$. Then, by [HHTU11] $m(\mathrm{Phc}^+(p_g) \cap \mathrm{Phc}^-(p_g)) = 1$ and g is ergodic. This proves f is stably ergodic.

Step 2 *If $g \in \mathcal{U}(f) \cap \mathrm{Diff}^2_m(M)$ then $W^u_g(p_g)$ is dense.*

Let $\omega_g(x)$ be the ω-limit set of $x \in M$. It is well known that $\omega_g(x)$ is a g-invariant closed set. By Poincaré's recurrence theorem, $x \in \omega_g(x)$ for m-a.e.$x \in M$, then $\overline{\mathcal{O}_g(x)} \subset \omega_g(x)$, but by ergodicity $\overline{\mathcal{O}_g(x)} = M$ for m-a.e.$x \in M$. Then $\omega_g(x) = M$ for m-a.e.$x \in M$.
By hypothesis $m(\mathrm{Phc}^-(p_g)) > 0$ then there exists $x \in \mathrm{Phc}^-(p_g)$ such that $\omega_g(x) = M$ and therefore $W^-(x) \pitchfork W^u(\mathcal{O}_g(p_g)) \neq \phi$. Let $y \in W^-(x) \pitchfork W^u(\mathcal{O}_g(p_g))$, it's easy to see that $\omega_g(y) = \omega_g(x) = M$ and then $\overline{W^u(\mathcal{O}_g(p_g))} = M = \overline{\mathcal{O}(W^u(p_g))}$.
Now, W^u is minimal then for all $f^k(p_f) \in \mathcal{O}_f(p_f)$ we have $W^-(f^k(p_f)) \pitchfork W^u(p_f) \neq \phi$ then there exists $\mathcal{U}(f)$ (we maintain the name) such that $\forall g \in \mathcal{U}(f) : W^-(g^k(p_g)) \pitchfork W^u(p_g) \neq \phi$.
Let's see that $W^u(g^k(p_g)) \subset \overline{W^u(p_g)}$, $\forall k \in \mathbb{Z}$
Let $l \in \mathbb{N}$ be the period of p_g and we consider $g^l : M \to M$. Let $D^u_k \subset W^u(g^k(p_g))$ a compact disk containing $g^k(p_g)$ and $D = W^u(p_g)$. Then by λ-lemma $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ there exists $D_n \subset D$ such that $d_{C^1}(g^{ln}(D_n), D^u_k) < \varepsilon$. Let $z \in W^u(g^k(p_g))$ then $\exists m \in \mathbb{N}$ such that $z' = g^{-lm}(z) \in D^u_k$. But, by the above, there exists a sequence $(z_n) \subset W^u(p_g)$ such that $g^{ln(z_n)}$ converges to z'. Therefore $z \in \overline{W^u(p_g)}$ and then $W^u(g^k(p_g)) \subset \overline{W^u(p_g)}$, $\forall k \in \mathbb{Z}$.
This implies $M = \overline{\mathcal{O}(W^u(p_g))} \subset \overline{W^u(p_g)}$, i.e. we obtain $W^u_g(p_g)$ is dense.

The previous step and the theorem 1.1.4 implies the minimality of W_g as we wanted to show. By Alexeev cone criterion for E^u_f and the integrability of the unstable bundle we have there exists a C^1-neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}^1_m(M)$ such that the map $g \mapsto TW_g$ is continuous on $\mathcal{U}(f)$. This show that W^u is stably minimal.
□

Remark 3.1.1 *If there is $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ such that the map $g \mapsto W_g$ is continuous, where W_g is a foliation, then $\{g \in \mathcal{U} : W_g \text{ is minimal}\}$ is a G_δ-set.*

Proof. Given $\varepsilon > 0$ and $L > 0$, the set $\mathcal{O}_{L,\varepsilon} = \{g \in \mathcal{U} : W^g_L(x) \,\mathrm{is}\, \varepsilon - \mathrm{dense}\}$ is an open set, then $G = \cap_{n\geq 1} \cap_{m \geq 1} \mathcal{O}_{m,1/n}$ is a G_δ-set and if $g \in G$ we have W_g is minimal. □

The following result is a weak version of the theorem 1.1.3.

Theorem 3.1.3 *Let $f \in \mathrm{Diff}^1_m(M)$, W_f an f-invariant expanding foliation such that:*

1. *There exists F an invariant bundle such that the splitting $TM = F \oplus_< TW_f$ is dominated.*
2. *there exists a hyperbolic periodic point p_f with unstable index $u = \dim W_f$ such that $\mathrm{Phc}^u(p_f) = M$.*

Then there exists $\mathcal{U}(f) \subset \mathrm{Diff}^1_m(M)$ such that:

(I) There exists $\mathcal{R} \subset \mathcal{U}(f)$ a residual set such that for all $g \in \mathcal{R}$ W_g is minimal.

(II) If $\dim(M) = 3$, there exists $\mathcal{R} \subset \mathcal{U}(f)$ a residual set such that all $g \in \mathcal{R}$ is stably Bernoulli and non-uniformly hyperbolic.

(III) If $\dim(M) = 3$, there exists $\mathcal{R} \subset \mathcal{U}(f)$ a residual set such that for all $g \in \mathcal{R}$ we have W_g is stably minimal.

To start the proof of the last theorem I will cite four key results. The following is a result of Jana Rodriguez Hertz.

Theorem 3.1.4 (Theorem 1.1, [Her12]) *Let M be a closed connected manifold of dimension 3, then there exists $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$ a residual set such that every $f \in \mathcal{R}$ satisfies one of the following alternatives:*

- *All Lyapunov exponents of f vanish almost everywhere, or*
- *f is ergodic and nonuniformly hyperbolic.*

the second is a result due to Bochi-Viana [BV05]

Theorem 3.1.5 (Theorem 1, [BV05]) *There exists a residual set $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$ such that, for each $f \in \mathcal{R}$ and m-almost every $x \in M$, the Oseledets' splitting of f is either trivial or dominated at x.*

the third is a result about of the continuity of the ergodic decomposition due to Ávila-Bochi [AB12].

Theorem 3.1.6 ([AB12]) *There exists a residual set $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$ such that for $f \in \mathcal{R}$ if there exists $p \in \mathrm{Per}_H(f)$ with $m(\mathrm{Phc}(p)) > 0$, then there exists a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^1_m(M)$ such that $m(\mathrm{Phc}(p_g)) > 0$ for all $g \in \mathcal{U}(f) \cap \mathrm{Diff}^2_m(M)$.*

and the last one is a result due to Abdenur-Bonatti-Crovisier [ABC11].

Theorem 3.1.7 ([ABC11]) *Given a generic $f \in \mathrm{Diff}^1_m(M)$ and μ a f-invariant ergodic measure. Then there exist a sequence of measures $\{\mu_n\}$, each supported on a periodic orbit, such that:*

(a) μ_n *converges to* μ *in the weak-star topology.*

(b) $\operatorname{supp}\mu_n$ *converges to* $\operatorname{supp}\mu$ *in the Hausdorff topology.*

(c) the Lyapunov exponents of f with respect to μ_n converge to the Lyapunov exponents with respect to μ.

Proof.[Proof of Theorem 3.1.3]

(I) By Alexeev cone criterion for E^u_f and the integrability of the unstable bundle we have there exists a C^1-neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}^1_m(M)$ such that the map $g \mapsto TW_g$ is continuous on $\mathcal{U}(f)$ and W_g is a g-invariant expanding foliation.
By hypothesis $\mathrm{Phc}^u(p_f) = M$, then by the corollary 3.1.2 $\mathrm{Phc}^u(p_g) = M$ in a C^1-neighborhood, which we still call $\mathcal{U}(f)$. By Bonatti-Crovisier (see [BC04], Theorem 1.3) there exists a residual set $\mathcal{R} \subset \mathcal{U}(f)$ such that $\overline{W^u(p)} = M$, for all $g \in \mathcal{R}$.
The previous argument, together with the Theorem 1.1.4 imply the minimality of W_g.

(II) By Theorem 3.1.4 [Her12] there exists a residual set $\mathcal{R} \subset \mathcal{U}(f)$ such that all $g \in \mathcal{R}$ is ergodic, non-uniformly hyperbolic and g has a dominated splitting $TM = E^-_g \oplus E^+_g$. By the ergodic decomposition theorem there exists a hyperbolic periodic point q_g of g with unstable index $u(q_g) = \dim E^+_g$ that satisfy $\mathrm{Phc}^+(q_g) \overset{\circ}{=} \mathrm{Phc}^-(q_g) \overset{\circ}{=} M$. Here $TW_g \subset E^+_g$ then $u(p_g) \leq u(q_g)$. The proof is divided into two cases.

Case 1 *The periodic points p_g and q_g have the same unstable index. Generically the periodic points with the same index are homoclinically related, then $\mathrm{Phc}^u(q_g) = \mathrm{Phc}^u(p_g)$ and $\mathrm{Phc}^-(q_g) = \mathrm{Phc}^-(p_g)$ (see, for instance [HHTU11]).*
By 3.1.6 [AB12] generically the ergodic decomposition is continuous, then $m(\mathrm{Phc}^-(q_g)) > 0$ in a neighborhood of g. Theorem 1.1.3 imply g is stably ergodic and non-uniformly hyperbolic. This proved (II) in this case.

Case 2 *The unstable indices are not equal, i.e.* $u(p_g) < u(q_g)$
If $u(p_g) < u(q_g)$ *then* $\dim W^u_g(x) = u(p_g) < u(q_g) = \dim E^+_g$. *By hypothesis the splitting* $TM = F \oplus_< TW_f$, *so we have a dominated splitting* $TM = F_g \oplus_< TW_g$ *in a neighborhood of* f. *By other side* $TM = E^-_g \oplus_< E^+_g$, *then we have a dominated splitting* $TM = E^g_1 \oplus_< E^g_2 \oplus_< E^g_3$, *where the extremal sub-bundle are one-dimensional and then* E^g_1 *and* E^g_3 *are hyperbolic. This show that* g *is partially hyperbolic and by [RHRHU08]* g *is generically stably ergodic. This complete the proof of (II).*

(III) We consider the residual set given by the previous item. As before, we will divide the proof in two cases.

Case 1 *The periodic points* p_g *and* q_g *have the same unstable index.*
By the same argument given in the first case in (II) we have W_g *is stably minimal.*

Case 2 *The unstable indices are not equal, i.e.* $u(p_g) = 1 < 2 = u(q_g)$
By item (I) generically g *has a one-dimensional expanding foliation* W_g *and* $\mathrm{Phc}(p_g) = M$.
If $u(p_g) + 1 = u(q_g)$ *then by theorem 2.8.1 of [HHTU10] we obtain an arbitrarily small* C^1*-perturbation of* g, *which admit a cu-blender associated with the analytic continuation of* q *(we mantein the names). This situation is* C^1*-robust, and by 2.8.1 there exists a* C^1*-neighborhood* $\mathcal{U}(g) \subset \mathrm{Diff}^1_m(M)$ *such that:*

$$W^+(q_h) \subset \overline{W^+(p_h)} \tag{3.1}$$

where p_h *and* q_h *are the analytic continuation of the points* p_g *and* q_g *respectly.*
By the minimality criterion 1.1.4, as $W^+(q_h)$ *is dense in* $\mathrm{Phc}(q_h)$ *we have* W_g *is stably minimal. This complete the proof of (III).*

□

3.2 Proof of Main Theorems

In this section we will to show the main theorems 1.1.1 and 1.1.2. As before, there exists $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$ a residual set such that for all $f \in \mathcal{R}$, we have that f is ergodic, non-uniformly hyperbolic and there exists a hyperbolic periodic point q_f such that $TM = E_f^- \oplus E_f^+$, $u(q_f) = \dim(E_f^+)$ and $\mathrm{Phc}(q_f) = M$. Suppose that f has a minimal expanding f- invariant foliation W_f, then $TW_f \subset E_f^+$. Again, we will divide the proof in two cases:

Case 1 *If $TW_f = E_f^+$ then by theorem 1.1.3 we have W_f is stably minimal and f is stably Bernoulli.*

Case 2 *If $TW_f \subsetneq E_f^+$ then $\dim(TW_f) = 1 < 2 = \dim(E_f^+)$. We will divide this case in two subcases:*

(i) Suppose that there exists p_f a hyperbolic periodic point of f with unstable index $u(p_f)$ equal to one.
By Theorem 3.1.5 [BV05] generically for m-almost every $x \in M$ the Oseledets splitting of f is either trivial or dominated at x. As f is ergodic for m-almost every $x \in M$ we have that the orbit of x is dense in the manifold M. This implies that the Oseledets splitting is dominated in the manifold M.
Let $\lambda_1 < 0 < \lambda_2 \leq \lambda_3$ the Lyapunov exponents of f. We claim that generically

$$\lambda_2 < \lambda_3$$

Indeed, if $\lambda_2 = \lambda_3$ then the Oseledets splitting (global) has the form $TM = F_1 \oplus_< F_2$, where $\dim(F_1) = 1$ and $\dim(F_2) = 2$. This splitting is the finest dominated splitting (because the exponents are equal) then by Theorem 3.1.7 [ABC11] generically there exists a sequence $\{p_n\}$ of periodic points such that the Lebesgue measure is approximate by periodic measures $\{\mu_n\}$ (each supported on the periodic orbit $\mathcal{O}(p_n)$) and the Lyapunov exponents of f with respect to μ_n converge to the exponents with respect to Lebesgue. Let $\mathcal{V}_1^n, \mathcal{V}_2^n, \mathcal{V}_3^n$ be the eigenvalues of $D_{p_n} f^{\pi(p_n)}$, where $\pi(p_n)$ is the period of p_n. Then $D_{p_n} f^{\pi(p_n)}(v_j^n) = \mathcal{V}_j^n v_j^n$ and $|\mathcal{V}_j^n| = e^{\lambda_j \pi(p_n)}$.
Then, if n is large enough we have $\mathcal{V}_2^n$ is close enough to $\mathcal{V}_3^n$, then making a small C^1-perturbation of f (conservative) we can suppose that f has a hyperbolic periodic point with complex eigenvalues. This situation is not possible, because in this case f does not admit a f-invariant expanding foliation one dimensional, then $\lambda_2 < \lambda_3$ as we wanted to show.
Thus generically the Oseledets splitting has the form $E_1 \oplus_< E_2 \oplus_< E_3$ and $TW_f = E_2$ or $TW_f = E_3$. Here the extremal sub-bundle E_1 and E_3 are

one-dimensional then f is partially hyperbolic and then by [RHRHU08] f is stably Bernoulli.

- *If $TW_f = E_2$, as the be an Anosov diffeomorphism is a open condition and the dominated splitting persist under C^1 perturbation, then by [Ham13] W_f is stably minimal.*
- *If $TW_f = E_3$, then as before the situation $u(p_f) + 1 = u(q_f)$ implies that generically (by theorem 2.8.1 - [HHTU10]) f admit a cu-blender associated with p_f, then there exists a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^1_m(M)$ such that:*

$$W^+(q_g) \subset \overline{W^+(p_g)}$$

where p_g and q_g are the analytic continuation of the points p_f and q_f respectly. By the minimality criterion 1.1.4, as $W^+(q_g)$ is dense in $\mathrm{Phc}(q_g)$ we have W_f is stably minimal.

(ii) If all hyperbolic periodic points have the same index then we have two possibilities:

- *There exists a C^1-neighborhood $\mathcal{U}(f)$ such that every $g \in \mathcal{U}(f)$ has not hyperbolic periodic points of different index (and then these points has the same index of the periodic points of f). Then by Theorem 2.6.3 f is an Anosov diffeomorphism. This implies that f is stably Bernoulli.*
  Here W_f is the strongest foliation of an Anosov diffeomorphism or TW_f is the intermedial subbundle of an Anosov diffeomorphism. In this last case by [Ham13] we have W_f is stably minimal.
- *In another case, making a C^1-perturbation of f, we can suppose that f has hyperbolic periodic points of different indices. This case we have already discussed previously.*

This completes the proof of the theorems 1.1.1 *and* 1.1.2*.*

Bibliography

[AB12] A. Avila and J. Bochi. Nonuniform hyperbolicity, global dominated splittings and generic properties of volume-preserving diffeomorphisms. *Transactions of the American Mathematical Society*, 364(6):2883–2907, jun 2012.

[ABC11] F. Abdenur, Ch. Bonatti, and S. Crovisier. Nonuniform hyperbolicity for $\mathcal{C}^1$-generic diffeomorphisms. *Israel Journal of Mathematics*, 183(1):1, Jun 2011.

[ACW16] A. Avila, S. Crovisier, and A. Wilkinson. Diffeomorphisms with positive metric entropy. *Publications mathématiques de l'IHÉS*, 124(1):319–347, oct 2016.

[ACW17] A. Avila, S. Crovisier, and A. Wilkinson. C^1 density of stable ergodicity. *ArXiv e-prints*, September 2017.

[Ano69] D. V. Anosov. Geodesic flows on closed riemann manifolds with negative curvature. *Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder American Mathematical Society, Providence, R.I.*, 124(90):iv+235, 1969.

[Arn01] M-C. Arnaud. Création de connexions en topologie $\mathcal{C}^1$. *Ergodic Theory and Dynamical Systems*, 21(02), mar 2001.

[AS67] D. V. Anosov and Y. G. Sinai. SOME SMOOTH ERGODIC SYSTEMS. *Russian Mathematical Surveys*, 22(5):103–167, oct 1967.

[BC04] Ch. Bonatti and S. Crovisier. Récurrence et généricité. *Inventiones mathematicae*, 158(1), apr 2004.

[BDP03] Ch. Bonatti, L. Díaz, and E. Pujals. A c1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources. *Annals of Mathematics*, 158(2):355–418, sep 2003.

[BDPR00] Ch. Bonatti, L.J. Díaz, E.R. Pujals, and J. Rocha. Robustly transitive sets and heterodimensional cycles, 2000.

[BDU02] Ch. Bonatti, L. Díaz, and R. Ures. Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms. *Journal de l'Institut de Mathematiques de Jussieu*, 1(4):157–193, 2002.

[BDV05] Ch. Bonatti, L. J. Díaz, and M. Viana. *Dynamics Beyond Uniform Hyperbolicity*. Springer-Verlag, 2005.

[BV00] Ch. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel Journal of Mathematics*, 115(1):157–193, dec 2000.

[BV05] J. Bochi and M. Viana. The lyapunov exponents of generic volume-preserving and symplectic maps. *Annals of Mathematics*, 161(3):1423–1485, may 2005.

[CP15] S. Crovisier and R. Potrie. Introduction to partially hyperbolic dynamics. 2015.

[GPS94] M. Grayson, C. Pugh, and M. Shub. Stably ergodic diffeomorphisms. *The Annals of Mathematics*, 140(2):295, sep 1994.

[Ham13] A. Hammerlindl. Leaf conjugacies on the torus. *Ergodic Theory and Dynamical Systems*, 33(03):896–933, jan 2013.

[Her12] M.A. Rodriguez Hertz. Genericity of nonuniform hyperbolicity in dimension 3. *Journal of Modern Dynamics*, 6(1):121–138, may 2012.

[HHTU10] F. Rodriguez Hertz, M.A. Rodriguez Hertz, A. Tahzibi, and R. Ures. Creation of blenders in the conservative setting. *Nonlinearity*, 23(2):211–223, jan 2010.

[HHTU11] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, and R. Ures. New criteria for ergodicity and nonuniform hyperbolicity. *Duke Mathematical Journal*, 160(3):599–629, dec 2011.

[Hop39] E. Hopf. Statistik der geodätischen linien in mannigfaltigkeiten negativer krümmung. *Berichten der Sächsischen Akademie der Wissenschaften zu Leipzi*, pages 261–304, 1939.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publications mathématiques de l'IHÉS*, 51(1):137–173, dec 1980.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[Mañ78] R. Mañé. Contributions to the stability conjecture. *Topology*, 17(4):383–396, 1978.

[Mañ82] R. Mañé. An ergodic closing lemma. *The Annals of Mathematics*, 116(3):503, nov 1982.

[Ose68] V. Oseledec. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Transactions of the Moscow Mathematical Society*, 19:197–221, 1968.

[Pal69] J. Palis. On morse-smale dynamical systems. *Topology*, 8(4):385–404, sep 1969.

[PdM78] J. Palis and W. de Melo. *Introdução aos sistemas dinâmicos*. Projeto Euclides. Instituto de Matemática Pura e Aplicada, 1978.

[Pes77] Ya B Pesin. Characteristic lyapunov exponents and smooth ergodic theory. *Russian Mathematical Surveys*, 32(4):55–114, aug 1977.

[PS96] C. Pugh and M. Shub. Stable ergodicity and partial hyperbolicity. In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 182–187. Longman, Harlow, 1996.

[RHRHU08] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle. *Inventiones Mathematicae*, 172(2):353–381, 2008.

[Rob70] R. Clark Robinson. Generic properties of conservative systems II. *American Journal of Mathematics*, 92(4):897, oct 1970.

[Sam14] M. Sambarino. A (short) survey on Dominated Splitting. *ArXiv e-prints*, March 2014.

[Tah04] Ali Tahzibi. Stably ergodic diffeomorphisms which are not partially hyperbolic. *Israel Journal of Mathematics*, 142(1):315–344, dec 2004.