MONOTONE MEASURES FOR DYNAMICAL SYSTEMS

Pablo Monzón *

* *Instituto de Ingeniería Eléctrica, Facultad de Ingeniería, Universidad de la República, Uruguay.*
e-mail: monzon@fing.edu.uy

Abstract*: In this work we state the relationships between almost global stability and monotone Borel measures for dynamical systems. We think that these ideas can help to understand density functions and almost global stability.*

Keywords: Nonlinear systems, density functions, almost global stability, planar systems.

1. INTRODUCTION

The almost global stability of dynamical systems is a concept weaker than global asymptotical stability but that can fit well in nonlinear control applications, specially when it is combined with local asymptotical stability. The concept and a sufficient condition for almost global stability were stated in the year 2001 by Anders Rantzer (Rantzer, 2001*b*) as a dual Lyapunov Method and has opened a new research direction in the nonlinear control field for both analysis and synthesis. The main result in (Rantzer, 2001*b*) is based on the existence of a density function, a kind of a dual of a Lyapunov function, that allows us to measure the growth of given sets along the flow.

In this work we explore the same ideas just in terms of measures defined over Borel sets of $\mathcal{R}^n$. We re-state the known results in this framework and we present new results. We think that this ideas based on measures instead of density functions can help to understand the structure of almost globally stable dynamics and the properties of density functions.

In Section 2 we state the basic definitions of almost global stability and density functions and the results between both concepts. In Section 3 we introduce the idea of monotone Borel measures and the relationships with almost global stability. We study some converse results in Section 4. Finally we present some conclusions.

2. DENSITY FUNCTIONS AND ALMOST GLOBAL STABILITY

In this Section we introduce the works of Anders Rantzer reported in several papers appeared in the last years, starting with the basic article (Rantzer, 2001*b*) and complemented with other related reports and publications (Prajna, 2004; **?**; Rantzer, 2001*a*; Angeli, 2004). We said that the origin is an almost global attractor (a.g.s.) if the complement of the set of points that are attracted to the origin has zero Lebesgue measure. For $x_0 \in \mathcal{R}^n$, let $f^t(x_0)$ denote the time t of the trajectory that starts at x_0. Then the system is a.g.s. if the set

$$\left\{ x \in \mathcal{R}^n \mid \lim_{t \to +\infty} f^t(x) \neq 0 \right\}$$

has zero Lebesgue measure. This concept of stability is weaker than the classical global asymptotic stability (g.a.s.) but can complement well the local asymptotical stability (l.a.s.) property. The key contribution of (Rantzer, 2001*b*) was the introduction of a particular kind of functions that for a.g.s. systems play a role similar to the Lyapunov functions for asymptotically stable systems: the density functions. Given a dynamical system $\dot{x} = f(x)$, a density function for this system is a scalar function $\rho : \mathcal{R}^n \setminus \{0\} \to [0, +\infty)$, of class C^1, integrable outside of a ball centered at the origin, and such that the following divergence condition is satisfied

$$\nabla.(\rho f)(x) > 0 \textit{ almost everywhere } (a.e.) \quad (1)$$

The result is based on a Liouville-like Lemma:

Lemma 2.1. Let $f \in C^1(D, \mathcal{R}^n)$ where $D \subset \mathcal{R}^n$ is open; consider the system $\dot{x} = f(x)$ and let $\rho \in C^1(D, \mathcal{R})$ be integrable. For a measurable set Z and a given time $t \geq 0$, assume that

$$f^\tau(Z) = \{f^\tau(x) \mid x \in Z\}$$

is a subset of D for all $0 \leq \tau \leq t$. Then

$$\int_{f^t(Z)} \rho(x)dx - \int_Z \rho(x)dx =$$

$$\int_0^t \int_{f^\tau(Z)} [\nabla.(f\rho](x)dxd\tau \quad (2)$$

When we can apply Lemma 2.1 to an invariant set ($f^t(Z) = Z$), we conclude that it has zero measure. Observe that if exists a density function, we can apply Lemma 2.1 to every Borel set Z such that $0 \notin \bar{Z}$. In particular, if there is a density function for the system, an invariant set with non zero Lebesgue measure must contain the origin in its closure.

In the same way, using ρ we can define a Borel measure μ compatible with the Lebesgue measure. If the μ-measure of a given set is preserved by the dynamical system (a particular case is when we are dealing with an invariant set) and if we can apply Lemma 2.1, then the sign definition of the divergence implies that the given set has zero Lebesgue measure.

3. MONOTONE MEASURES

In the previous Section we have seen how the existence of a density function implies the almost global stability of the system. Using the density function we can construct a measure over $\mathcal{R}^n$ that grows along the trajectories, due to the sign definition of the divergence, and is finite for sets that can be isolated from $\{0\}$, due to the integrability condition. Actually, this are the meanings of the identity (2), the divergence condition (1) and the integrability requirement. So we can re-state Theorem 1 in (Rantzer, 2001*b*) just in terms of measures as follows.

Theorem 3.1. Given the equation $\dot{x} = f(x)$ where $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, $f(0) = 0$ and $\frac{\|f(x)\|}{\|x\|}$ is globally bounded, suppose there exists a Borel measure μ in $\mathcal{R}^n$, such that:
• $\mu[B^c(0, \epsilon)] < +\infty$ for every $\epsilon > 0$.
• for every Borel set Y with $0 < \mu(Y) < +\infty$ and for every $t > 0$

$$\mu\left[f^t(Y)\right] > \mu(Y) \quad (3)$$

Then the origin is almost globally stable.

We want to emphasize that this approach can drive us to a new set of results that we want to explore. Inequality (3) is crucial and this kind of behavior will be used along the article, so we introduce the following definition.

Definition 3.1. Given a vector field $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, a Borel measure μ is said to be monotone if for all Borel set Y it is true that if $\mu(Y) = 0$ then $\lambda(Y) = 0$ (being λ the Lebesgue measure) and for every non-zero and finite μ-measure set Y

$$\mu\left[f^t(Y)\right] - \mu(Y)$$

has definite sign for all $t > 0$.

$\diamondsuit$

Observe that from the above definition, the measure of any invariant set, that is, a set Y satisfying $f^t(Y) = Y$ for all t, must be 0 or $+\infty$. As a consequence, the measure of the whole space should be infinite. In the same way, if for a given μ-measurable set Y with finite measure there exists $t > 0$ such that

$$\mu\left[f^t(Y)\right] = \mu(Y)$$

then $\mu(Y) = 0$ and Y has zero Lebesgue measure. We will deal with two particular kinds of monotone Borel measures: the ones *bounded over compact sets* and the ones *bounded at infinity* (for every $\epsilon > 0$, the exterior of the ball of radius ϵ centered at the origin has finite measure μ). The previous definitions can be extended for systems on a manifold in a direct way.

It is clear that every density function for a given system induces a growing measure bounded at infinity. For examples of density functions we refer the reader to (Rantzer, 2001*b*; Prajna, 2004). A decreasing measure bounded on compact sets recovers the idea that every set *shrinks to the attractor*. We present an illustrative example.

Example 3.1. Consider the two dimensional torus and the system described by the equations

$$\begin{aligned}\dot{\Phi}_1 &= \sin(\Phi_2) - \sin(\Phi_1) \\ && \Leftrightarrow \dot{\Phi} = F(\Phi) \\ \dot{\Phi}_2 &= \sin(\Phi_1) - \sin(\Phi_2)\end{aligned}$$

with $\Phi_1 + \Phi_2 = 2\pi$, which is a particular case of two coupled oscillators that appears in some biological systems (Strogatz, 2000). Consider the functions

$$\rho(\Phi) = \frac{1}{1-\cos(\Phi_1)}$$

$$l(\Phi) = \frac{1}{1+\cos(\Phi_1)}$$

It follows that

$$\nabla.\left[\rho.F\right](\Phi) = \frac{2}{1-\cos(\Phi_1)}$$

$$\nabla.\left[l.F\right](\Phi) = -\frac{2}{1+\cos(\Phi_1)}$$

Then ρ and l induce respectively an increasing and a decreasing measure.

△

4. CONVERSE RESULTS FOR ALMOST GLOBAL STABILITY

In this section it is shown that almost global stability implies the existence of a monotone measure as the one in Theorem 3.1.

As was shown in (Monzón, 2003*b*), every linear Hurwitz [1] system admits a density function of the form

$$\rho(x) = \frac{1}{\left[V(x)\right]^{\alpha}}$$

where V is a quadratic Lyapunov function for the linear system and α should be chosen big enough in order to satisfy both the integrability condition and inequality (1).

We present here two useful Lemmas which are related to the constructions presented in (Monzón, 2003*b*). A result similar to Lemma 4.1 can be found in (Wirth, 1999) in a different context. We will use the auxiliar canonical asymptotically stable linear system

$$\dot{y} = g(y) = -y \tag{4}$$

Lemma 4.1. Consider the system

$$\dot{x} = f(x) \tag{5}$$

with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ and $x = 0$ being an asymptotically stable equilibrium point, such that

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

is a Hurwitz matrix. Consider also the linear system (4). Denote by R the open subset of $\mathcal{R}^n$ which is the region of attraction of the origin. Then there exists a continuous function $h_1 : R \to \mathcal{R}^n$, satisfying that for every $x \in R$ and every τ such that $f^t(x)$ is defined, the following is true

$$h_1 \circ f^{\tau}(x) = g^{\tau} \circ h_1(x) \tag{6}$$

Moreover, if the system is complete (the trajectory through every point is defined for every real t, i.e. no finite escape time), then h_1 is an homeomorphism.

Proof: Again $f^t(x_0)$ will denote the trajectory at time t for system (5), starting at x_0. In the same way, $g^t(y_0)$ will refer to the trajectories of the linear system (4).

Since A is a Hurwitz matrix, the nonlinear system admits a quadratic local Lyapunov function of the form $\mathcal{V}(x) = x^T P x$, with $P = P^T > 0$ and $A^T P + PA < 0$. Consider the ellipsoid

$$\mathcal{E} = \left\{x \in R \mid x^T P x = \delta\right\}$$

with δ small enough such that $\mathcal{E}$ is included in the domain of definition of $\mathcal{V}$. It must be clear that all the trajectories of the region of attraction of the origin intersect the ellipsoid just once, since they converge to the origin and the ellipsoid is a level curve of the function $\mathcal{V}$, which decreases along the trajectories. Let $H : \mathcal{R}^n \to \mathcal{R}^n$ be a C^∞ diffeormophism carrying the ellipsoid $\mathcal{E}$ to the sphere

$$\mathcal{S} = \{y \in \mathcal{R}^n \mid \|y\| = 1\}$$

Function H can be taken in a way such that the orientation of those manifolds is preserved, i.e.,

$$det\left[\frac{\partial H}{\partial x}(x)\right] > 0$$

For every point $x \in R$ define $t(x)$ as the time corresponding to the intersection of the trajectory through x with the ellipsoid, that is $f^{t(x)}(x) \in \mathcal{E}$. We define $h_1 : R \to \mathcal{R}^n$ as follows

$$h_1(x) = g^{-t(x)}\left[H\left(f^{t(x)}(x)\right)\right]$$

Figure 1 shows the construction process for h_1. For $x = 0$, we put $h(0) = 0$. Points in the interior of the ellipsoid must flow *to the past* to reach it, having a corresponding negative time $t(x)$. Consider a given point x and a given time τ such that $f^{\tau}(x)$ exists. Then we have that

$$t\left[f^{\tau}(x)\right] = t(x) - \tau$$

[1] It represents a general asymptotically stable linear system $\dot{y} = Ay$ with A Hurwitz, i.e. all the eigenvalues of A lying in the open left half complex plane.

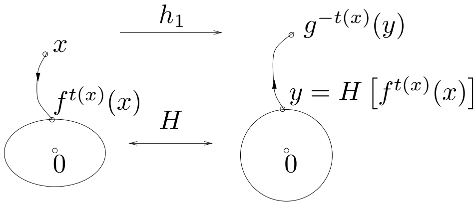

Figure 1. *Definition of function h_1.*

So

$$h_1\left[f^{\tau}(x)\right] = g^{-t[f^{\tau}(x)]}\left[H\left[f^{t[f^{\tau}(x)]}\left(f^{\tau}(x)\right)\right]\right]$$

$$= g^{-t(x)+\tau}\left[H\left(f^{t(x)}(x)\right)\right] = g^{\tau}\left[h_1(x)\right]$$

We can express this as follows

$$h_1 \circ f^{\tau}(x) = g^{\tau} \circ h_1(x)$$

for all $x \in R$, for all $\tau \in \mathcal{R}$ such that $f^{\tau}(x)$ exists.

By construction, h_1 is an open function, i.e. the image of an open set is also open. Note that the hole process is reversible, so the inverse of h_1 exists and is continuous. If the trajectories are defined for all real t, then the function h_1 is a homeomorphism between the region of attraction R and $\mathcal{R}^n$ and it is a continuous conjugacy between the nonlinear and the linear system.

■

Lemma 4.2. Consider the system (5) with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ and $x = 0$ being an asymptotically stable equilibrium point with region of attraction R, such that the Jacobian matrix at the origin is a Hurwitz matrix. Consider also the linear system (4). Then there exists a continuous function $h_2 : R \setminus \{0\} \to \mathcal{R}^n$, satisfying that for every non zero $x \in R$ and every τ such that $f^{\tau}(x)$ is defined, the following is true

$$h_2 \circ f^{\tau}(x) = g^{-\tau} \circ h_2(x) \tag{7}$$

Proof: Like in the previous Lemma, we obtain a Lyapunov level surface from the local exponential stability hypothesis. Then we define

$$h_2(x) = g^{t(x)}\left[H\left(f^{t(x)}(x)\right)\right] \tag{8}$$

Figure 2 shows the construction process for h_2. Observe that in this case, we move forward in time after we change from the nonlinear system to the linear one. The following facts are true. Their proofs are like in Lemma 4.1.

- The exterior of the ellipsoid $\mathcal{E}$ is mapped in the interior of the sphere $\mathcal{S}$;
- More general, the outside of a ball centered at the origin is mapped into the inside of a ball centered at the origin;
- For every $x \in R$ and for every $\tau \in \mathcal{R}$ such that $f^{\tau}(x)$ is defined $h_2 \circ f^{\tau}(x) = g^{-\tau} \circ h_2(x)$.

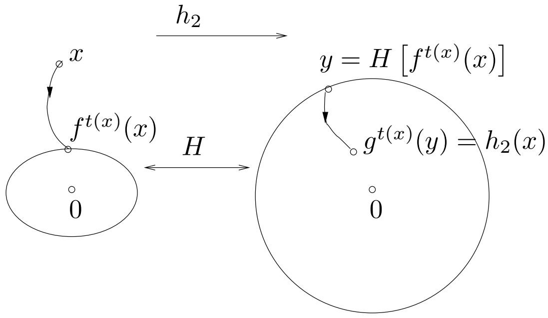

Figure 2. *Definition of function h_2.*

■

Remarks:

1) The previous lemmas give us a way to map the trajectories of the region of attraction of the nonlinear system into the trajectories of the linear one, with the possible exception of the trajectory through the origin. The function h_1 is a *time-preserving* correspondence and h_2 is a *time-reversing* one.

2) The function h_1 is as differentiable as the field f and this is an important fact as we will mention later. Moreover, it satisfies the following condition

$$det\left[\frac{\partial h_1}{\partial x}(x)\right] > 0 \quad \forall x \neq 0$$

since the flows preserve the orientation too. This is also true for h_2, for $x \neq 0$.

3) Local exponential stability of the origin is used only to obtain the ellipsoid $\mathcal{E}$, which is a surface diffeormorphic to the unit sphere $\mathcal{S}$. It can be replaced by any level surface of a local Lyapunov function, as long as it can be proved that this level surface is homeomorphic to the sphere. The existence of a Lyapunov function for an asymptotically stable system is ensured by Massera's result (Massera, 1949). The fact that a compact level surface is diffeomorphic to the unit sphere is true for surfaces of dimensions 1 and 2 and is guaranteed by the h-Cobordism Theorem of Smale (Smale, 1962; Milnor, 1965) for surfaces of dimensions equal or greater than 5, while for dimension 4 only an homeomorphism can be ensured (Freedman, 1982). So the hypothesis of the Lemmas 4.1 and 4.2 can be relaxed, asking for the origin to be a local attractor with no particular restriction on its linear approximation, and requiring that the dimension of the space be different from 4 (i.e. the Lyapunov level surface should have dimension different from 3).

We will use functions h_1 and h_2 to carry some properties of the linear system into the nonlinear system. Now we can prove the main result of this section.

Proposition 4.1. Consider the system (5) with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ and $x = 0$ being an almost globally asymptotically stable fixed point, such that the Jacobian matrix at the origin is a Hurwitz matrix. Then

(1) there exists a increasing Borel measure μ bounded at infinity.
(2) there exists a decreasing Borel measure finite on compacts.

Proof: We can proof the result either with the auxiliary functions h_1 or h_2. Observe that the existence of this functions is guaranteed by hypothesis. The process is the following: we will define a monotone Borel measure for the nonlinear system using a monotone Borel measure for the linear system.

Let us first use h_1. The domain of h_1 is almost all the space, due to the almost global stability property of the system. Let R be the region of attraction of the origin. It is open, invariant and connected (Khalil, 1996). Then, every Borel set $Y \subset \mathcal{R}^n$ can be split into two sets:

$$Y_R = Y \cap R \quad , \quad Y_{R^c} = Y \cap R^c$$

Observe that for every $t \geq 0$ such that $f^t(Y)$ exists,

$$\left[f^t(Y)\right]_R = f^t(Y_R) \quad , \quad \left[f^t(Y)\right]_{R^c} = f^t\left(Y_{R^c}\right)$$

Consider a scalar function $\sigma \in C^1(\mathcal{R}^n \setminus \{0\}, [0, +\infty))$. For a given Borel set $Y \subset \mathcal{R}^n$ define

$$\mu(Y) = \int_{h_1(Y_R)} \sigma(y) dy \tag{9}$$

It is clear that μ is a Borel measure, since R^c has zero Lebesgue and h_1 is one to one.

Consider a given time t such that $f^t(x)$ exists for every $x \in Y$. Then $\mu\left[f^t(Y)\right] - \mu(Y) =$

$$= \int_{h_1[f^t(Y_R)]} \sigma(y) dy - \int_{h_1(Y_R)} \sigma(y) dy$$

Using (6) we obtain $\mu\left[f^t(Y)\right] - \mu(Y) =$

$$= \int_{g^t[h_1(Y_R)]} \sigma(y) dy - \int_{h_1(Y_R)} \sigma(y) dy$$

And if we can apply Lemma 2.1, $\mu\left[f^t(Y)\right] - \mu(Y) =$

$$= \int_0^t \int_{g^\tau[h_1(Y_R)]} \nabla . \left[\sigma . g\right](y) dy d\tau$$

If σ is a density function for the linear field, and if we can apply Lemma 2.1, we see that μ is an increasing measure. Moreover, consider an arbitrary $\epsilon > 0$ and assume that $Y \subset B^c(0, \epsilon)$. Then

$$\mu(Y) = \int_{h_1(Y_R)} \sigma(y) dy < +\infty$$

since h_1 is an open function and σ is integrable outside arbitrary neighborhoods of the origin. So μ is a monotone Borel measure bounded at infinity.

If we choose σ such that $\nabla . \left[\sigma . g\right](y) < 0 \quad a.e.$, we conclude that μ is a decreasing measure, and if σ is integrable over compacts sets, μ turns out to be bounded over compact sets. It is enough to take σ as a Lyapunov function since

$$\nabla . \left[\sigma . g\right](y) = \dot{\sigma}(y) + \nabla . g(y) . \sigma(y) = \dot{\sigma}(y) - n . \sigma(y)$$

We can repeat the previous arguments using function h_2. We only show the construction of an increasing measure. As before

$$\mu(Y) = \int_{h_2(Y_R)} \sigma(y) dy$$

If we can apply Lemma 2.1, $\mu\left[f^t(Y)\right] - \mu(Y) =$

$$\int_{g^{-t}[h_2(Y_R)]} \sigma(y) dy - \int_{h_2(Y_R)} \sigma(y) dy$$

$$= - \int_0^t \int_{g^{-\tau}[h_2(Y_R)]} \nabla . \left[\sigma . g\right](y) dy d\tau$$

due to (7). So if $\nabla . \left[\sigma . g\right] < 0$ almost everywhere and σ is integrable over compact sets, then μ is a monotone Borel measure bounded over compacts. If σ is a Lyapunov function for the linear system, μ is an increasing measure bounded at infinite (this last assertion comes from the reversing time property of h_2).

■

Due to the remarks mentioned after Lemmas 4.1 and 4.2 we can write this more general result:

Theorem 4.1. Let the system $\dot{x} = f(x)$ with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, $n \neq 4$, and $x = 0$ an almost globally asymptotically stable equilibrium point with local stability. Then

(1) there exists a monotone Borel measure μ bounded at infinity.
(2) there exists a monotone Borel measure finite on compacts.

□

Remarks:

4) Another observation we want to make is that if the field f is twice differentiable, we get that h_1 is twice differentiable too. Then, for the globally asymptotically stable case, the monotone Borel measure μ bounded over compacts comes from a density function $\bar{\rho}$ defined as

$$\bar{\rho}(x) = \sigma\left[h_1(x)\right] . \left|\frac{\partial h_1}{\partial x}(x)\right|$$

with σ a density function for the linear field (see (Monzón, 2003*b*) for details).

5. CONCLUSIONS

In this work we have introduced the concept of monotone Borel measures in the context of almost global stability of dynamical systems. We have shown that for an almost global asymptotical system we can find a monotone Borel measure (increasing or decreasing) and this idea complements the direct result of (Rantzer, 2001*b*) and the particular converse result of (Monzón, 2003*b*). We think that this approach can guide us to a new set of results, like the ones presented in (Monzón, 2003*a*).

6. ACKNOWLEDGEMENTS

I want to specially thank Professor Jorge Lewowicz for his time, dedication and advice.

REFERENCES

Angeli, D. (2004). An almost global notion of input-to-state stability. *Trans. on Automatic Control* **47**(3), 410–421.

Freedman, M.H. (1982). The topology of the four-dimensional manifolds. *J. of Diff. Geom.* **17**, 357–453.

Khalil, Hassan (1996). *Nonlinear Systems.* Prentice-Hall.

Massera, Jose Luis (1949). On Liapunoff's conditions of stability. *Annals of Mathematics* **50**, 705–721.

Milnor, John (1965). *Lectures on the h-Cobordism Theorem.* Princeton University Press.

Monzón, Pablo (2003*a*). Monotone measures and almost global stability of dynamical systems. Pre-Publicaciones Matemáticas de la Udelar. 2003/73.

Monzón, Pablo (2003*b*). On necessary conditions for almost global stability. *IEEE Trans. Automatic Control* **48**(4), 631–634.

Prajna, S.; Parrilo, P.A.; Rantzer A. (2004). Nonlinear control synthesis by convex optimization. *Trans. on Automatic COntrol* **41**(5), 310–314.

Rantzer, Anders (2001*a*). Almost global stability of phase-locked loops.. Technical report. Lund Institute of Technology.

Rantzer, Anders (2001*b*). A dual to Lyapunov's stability theorem. *Systems and Control Letters* **42**(3), 161–168.

Smale, S. (1962). On the structure of manifolds. *Amer. J. of Math.* **84**, 387–399.

Strogatz, S. H.; (2000). From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled nonlinear oscillators. *Physica D*, **143**, 1–20.

Wirth F., Grune L.; Sontag E.;(1999). Asymptotic stability equals exponential stability, and ISS equals finite energy gain - if you twist your eyes. *Systems and Control Letters* (38), 127–134.