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Structures hyperboliques et propriétés robustes des flots singuliers.

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To Perla Carrion, I won't ever let go...

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Résumé

S'il existe un ensemble ouvert dans la topologie C^r qui satisfait une certaine propriété, alors on dit éque la propriété est C^r-robuste. Tant pour diféomorphismes comme pour flots non singuliers, on a beaucoup de résultats à propos de porpriétés C^1-robustes et structures globales de la dynamique, par exemple l'hyperbolicité, l'hyperbolicité partielle ou les splittings dominés. En outre, plusieures difficultées se présentent lorsqu'une propriété robuste est satisfaite pour un ensemble d'orbites contenant orbites regulières qui s'accumulent contre des singularitées. Ce phénomène est compris surtout en dimension 3, mais jusqu'à présent il était un problème généraliser ce genre de résultats pour dimensions majeures.

Ce travail construit, en première instance, un ouvert de examples en dimensión 5 d'un flot étoile qui contient 2 singularitées avec diffrents indice, robustemment dans la même classe de récurrence par chaînes. Celà nous permet montrer qu'une généralisation directe des résultats qu'on a en dimmension 3 ne saurait être posibleê en dimensions plus hautes. En effet, on a des ensembles ouverts de flots étoile qui ne sont pas robustemment transitifs au sens classique.

Dans une deuxième instance, avec Christian Bonatti, on propose un procédé général pour adapter les structures hyperboliques usuelles aux singulières. On croit que cette interprétation de'effet des singularitées sur les structures hyperboliques, ouvre un chemin pour traiter avec la difficulté de la coéxistance robuste de singularitées et d'orbites régulières. En particulierm cette nouvelle définition nous permet de généraliser la démontration de [MPP] pour obtenir une caractérisation des flots étoile dans un ouvert dense en toute dimension.

Finalement, en utilisant la même stratégie mentionnée préécédemment, on récupère les résultats de [ABC] et [BDP] pour des flots. On montre il y a un ouvert et dense des flots dans le quel un flot avec une classe de récurrence robuste a un type d'hyperbolicité faible. Ceci montre que la manière qu'on propose pour interpréter les singularitées a le potentiel de s'adapter aux diverses situations dans lesquelles coéxistent les singularitées avec les orbites régularitéesé avec l'objectif de retrouver les résultats pour des difféomorphismes.

Resumen

Una propiedad de un sistema dinámico es C^r- robusta si se cumple para un conjunto abierto de sistemas con la topología C^r. Para difeomorfismos o flujo no singulares, existen muchos resultados relacionando propiedades C^1-robustas y estructuras globales de la dinámicas, como la hiperbolicidad, hiperbolicidad parcial o splittings dominados. Por otro lado existen dificultades cuando una propiedad robusta se cumple en un conjunto de órbitas conteniendo órbitas regulares que acumulan contra singularidades. Este fenómeno está bien entendido principalmente en dimensión 3, pero hasta ahora seguía siendo una obstrucción para generalizar este tipo de resultados en dimensiones más altas.

En este trabajo en primer lugar construimos un avierto de ejemplos en dimensión 5 de un flujo estrella que contiene 2 singularidades de distinto índice, robustamente en la misma clase de recurrencia por cadenas. Esto nos permite mostrar que una generalización directa de los resultados en dimensión 3, no va a ser posible en dimensiones más altas, es decir, existen conjuntos abiertos de flujos estrella, que no son singularmente hiperbólicos en el sentido clásico.

En segundo lugar, con Chrsitian Bonatti, proponemos un procedimiento general para adaptar las estructuras hiperbólicas usuales a las singularidades. Creemos que esta interpretación del efecto de las singularidades sobre las estructuras hiperbólicas, abre un camino para tratar con la ya mencionada dificultad de la coexistencia robusta de singularidades y órbitas regulares. En particular esta nueva definición nos permite generalizar la prueba en [MPP] para obtener una caracterización de los flujos estrella en un abierto y denso y para cualquier dimensión.

En tercer lugar, usando la misma herramienta mencionada arriba recuperamos los resultados en [ABC] y[BDP] para flujos. Mostramos hay un avierto y denso C^1 de campos en el que un flujo con una clase de recurrencia robusta tiene una forma de hiperbolicidad débil. Esto muestra que la manera que proponemos de interpretar las singularidades tiene el potencial de adaptarse a las diversas situaciones en las que coexisten singularidades y órbitas regulares con el fin de re obtener los resultados para difeomorfismos.

Abstract

A property of a dynamical system is called C^r-robust if it holds on a C^r-open set of systems. For diffeomorphisms or for non-singular flows, there are many results relating C^1-robust properties and global structures of the dynamics, as hyperbolicity, partial hyperbolicity, dominated splitting. However, a difículty appears when a robust property of a ow holds on a set containing recurrent orbits accumulating a singular point. This phenomenon is mainly understood in dimension 3, but till now it remained the main obstruction in order to recover these kind of results for singular flows

First, construct a robust example in a 5 dimensional manifold, of a a star flow containing 2 singularities of different indexes in the same chain recurrence class. This allows us to show that a direct generalization of the results in dimension 3 for singular star flows is not possible. i.e. There are open sets of star flows that are not singular hyperbolic in the classical sense.

Secondly, with Christian Bonatti we propose a a general procedure for adapting the usual hyperbolic structures to the singularities, opening the door for bypassing the difficulty of the coexistence of singular and regular orbits. In particular, this new definition allows us to adapt the proof in [MPP] to get a characterization of star flows on a C^1-open and dense set.

And third , using the same tool described above,we partially recover the results in [ABC] and [BDP] for flows, showing that there is a C^1-open and dense set of vector fields such that a flow having a robustly chain transitive sets has a weak form of hyperbolicity. This shows that the way we propose to interpret the effect of singularities, has the potential to adapt to other settings in which there is coexistence of singularities and regular orbits with the goal of reobtaining the results that we already know for diffeomorphisms.

Contents

Chapter 1

Introduction

1.1 General setting and historical presentation

Considering the infinite diversity of the dynamical behaviors, it is natural to have a special interest on the *robust properties* that is, properties that are impossible to break by small perturbations of a system; in other words, a dynamical property is robust if its holds on a (non-empty) open set of diffeomorphisms or flows.

1.2 Robust properties and dynamical structures for diffeomorphisms

1.2.1 Trapping regions

One of the oldest and very simple geometric idea for structuring the dynamics of a homeomorphism f is the use of *trapping* (or *attracting*) *regions*, that is a compact set U whose image $f(U)$ is contained in the interior of U. The *maximal invariant set* $\bigcap_{n\in\mathbb{Z}} f^n(U)$ *of* f *in* U is a compact invariant set called *the attracting set in* U. One important property of trapping regions is that they are C^0*-robust*: U is a trapping region for any homeomorphism g close to f in the C^0-topology.

Conley theory, [Co], pushes forward this simple idea for giving a general description of the dynamics of homeomorphisms:

— a point x is *chain recurrent* if, for any $\varepsilon > 0$, there is an ε*-pseudo orbit* from x to x, that is, a sequence $x = x_0, x_1 \dots, x_k = x$, $k > 0$ with $d(x_i, f(x_{i-1})) < \varepsilon$, for $i \in \{1, \dots, k\}$. He proves that x is chain recurrent if and only if, for any trapping region U, the orbit of x is either disjoint from U or contained in it.

— the set $\mathcal{R}(f)$ of all chain recurrent points is a compact invariant set called the *chain recurrent set*.

— two points x, y in $\mathcal{R}(f)$ are in the same *chain recurrence class* if there are ε-pseudo orbits from x to y and from y to x for any $\varepsilon > 0$. The chain recurrence classes are disjoint compact invariant sets. Two chain recurrent points x and y belong to the same class if and only if any trapping region containing one of the points x and y contains both of them.

— an important consequence is that any chain recurrence class admits a basis of neighborhoods which are *filtrating sets*, that is, intersection of an attracting region and of a *repelling region* (i.e. attracting region for the inverse f^{-1}).

1.2.2 Dominated splitting

The notion of trapping region, applied to the differential of a diffeomorphism, leads to the weakest notion of hyperbolic structure, introduced by Mañé and Liao called *dominated splitting*:

Definition 1. Let $f \colon M \to M$ be a diffeomorphism of a Riemannian manifold M and $K \subset M$ a compact invariant set of f, that is $f(K) = K$. A splitting $T_xM = E(x) \oplus F(x)$, for $x \in K$, is called dominated if

— $dim(E(x))$ is independent of $x \in K$ and this dimension is called the s-index of the splitting;

— it is Df-invariant: $E(f(x)) = Df(E(x))$ and $F(f(x)) = Df(F(x))$ for every $x \in K$;

— there is $n > 0$ so that for every x in K and every unit vectors $u \in E(x)$ and $v \in F(x)$ one has

$$\|Df^n(u)\| \leq \frac{1}{2}\|Df^n(v)\|.$$

One denotes $TM|_K = E \oplus_< F$ the dominated splitting.

Some of the most important properties of the domination, are: (see fore instance [BDV, Appendix B]):

— the bundles of a dominated splitting are *continuous*.

— given i there is a *unique* dominated spitting of s-index i

— if K admits a dominated splitting of s-index i then there is a neighborhood V of K so that for every diffeomorphisms g C^1-close to f, the maximal invariant set $\Lambda_g = \Lambda(g, V)$ admits a dominated spitting $TM_{\Lambda(g,V)}$ of s-index i.

The dominated spitting is unique if one fixes its dimensions. However, there can exist several dominated splittings on the same invariant compact set K and [BDP] proves the existence of a unique *finest dominated splitting*

$$TM|_K = E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_K$$

so that for every i the splitting

$$TM|_K = \bigoplus_1^i E_j \oplus \bigoplus_{i+1}^k Ej$$

is dominated, and every dominated splitting $E \oplus_< F$ is of this form.

1.2.3 Hyperbolic structures

Given a compact invariant set K of a diffeomorphism f a *(weak) hyperbolic structure* on K is

— a dominated splitting $TM|_K = E_1 \oplus_< \cdots \oplus_< E_k$

— for some $i \in \{1, \dots, k\}$ one requires the uniform expansion (or contraction) of some quantity related to the action of Df on the vectors in E_i.

Examples:

— K is *hyperbolic* if there is a dominated splitting $E^s \oplus E^u$ so that the vectors are uniformly contracted in E^s and uniformly expanded in E^u;

— K is *partially hyperbolic* if there a dominated splitting $TM|_K = E_1 \oplus_< \cdots \oplus_< E_k$ satisfying at least one of the two properties below:

  — the vectors in E_1 are uniformly contracted,

  — the vectors in E_k are uniformly expanded.

— K is *strongly partially hyperbolic* if there is a dominated splitting $E^s \oplus E^c \oplus E^u$ so that the vectors in E^s are uniformly contracted and the vectors in E^u are uniformly expanded

— K is *volume partial hyperbolic* if there is a dominated splitting $E^{cs} \oplus E^c \oplus E^{cu}$ so that the volume in E^{cs} is uniformly contracted and the volume in E^{cu} is uniformly expanded.

1.2.4 Structures and robustness of dynamical properties

One important starting point of the dynamical systems as a mathematical field has been the characterization of the structural stability (i.e. systems whose topological dynamics are unchanged under small perturbations) by the hyperbolicity (i.e. a global structure expressed in terms of transversality and of uniform expansion and contraction). This characterization, first stated by the stability conjecture [PaSm], was proven for diffeomorphisms in the C^1 topology by Robin and Robinson in [R1], [R2] (hyperbolic systems are structurally stable) and Mañé [Ma2] (structurally stable systems are hyperbolic). The equivalent result for flows (also for the C^1 topology) leads to extra difficulties and was proven by [H2].

We can see in this case how the robustness of the properties is related with the structure in the tangent space: in this case, a very strong robust property is related to a very strong uniform structure. However, hyperbolic systems are not dense in the set of diffeomorphisms or flows; instability and non-hyperbolicity may be robust. In order to describe a larger set of systems, one is lead to consider less rigid robust properties, and to try to characterize them by (weaker) structures that limit the effect of the small perturbations.

In this spirit there are several results for diffeomorphisms in the C^1 topology:

1. [Ma] proves that robustly transitive surface diffeomorphisms are globally hyperbolic (i.e. are Anosov diffeomorphism). This is no more true in higher dimensions (see examples in [Sh, Ma1]).
2. [DPU, BDP] show that robustly transitive diffeomorphisms admits a dominated splitting, and their finest dominated splitting is volume partially hyperbolic. This result extends to *robustly transitive sets*, and to *robustly chain recurrent sets*.
3. One says that a system is *star* if all periodic orbits are hyperbolic in a robust fashion: every periodic orbit of every C^1-close system is hyperbolic. For a diffeomorphism, to be star is equivalent to be hyperbolic (an important step is done in [Ma] and has been completed in [H]).

1.2.5 Lack of any structure, lack of any robust properties!

A long sequence of papers, starting with the work [Ma] of Mañé, and then [DPU, BDP, BDV], show that the dominated splittings is the unique obstruction for mixing the Lyapunov exponents of periodic orbits, by C^1-small perturbation of the diffeomorphism. We present informally in this section the most complete result in this spirit.

Let f be a diffeomorphism of a compact manifold of dimension d and let K be a compact set which is the Hausdorff limit of a sequence γ_n of periodic orbits whose periods tend to infinity. Assume that

$$TM|_K = E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_K$$

is the finest dominated splitting on K, and let $d_i = dimE_i$.

Let says that a matrix $A = (\alpha_{s,t}) \in M(d, \mathbb{R})$ is *compatible with the splitting* if

— $\alpha_{s,t} \geq 0$ for every s, t;

— the sum of the $\alpha_{s,t}$ in each line and in each column is equal to 1;

— the $\alpha_{s,t}$ vanish out of the diagonal blocks corresponding to the E_i, that is: if $\alpha_{s,t} \neq 0$ then there is $i \in \{1, \ldots, k\}$ so that $\sum_1^{i-1} d_j < s, t \leq \sum_1^i d_j$.

Theorem 1 ([BB]). *With the notations above, given any $\varepsilon > 0$, there is n_ε such that, for any $n \geq n_\varepsilon$, for any matrix $A = (\alpha_{s,t}) \in M(d, \mathbb{R})$ compatible with the splitting, there is a diffeomorphism g satisfying:*

— *g is ε-C^1-close to f;*
— *g coincides with f on γ_n and out of an arbitrarily small neighborhood of γ_n;*
— *the Lyapunov exponents $\lambda_{1,g} \leq \cdots \leq \lambda_{d,g}$ of γ_n for g satisfy*

$$\lambda_{i,g} = \sum \alpha_{i,t} \lambda_{t,f}.$$

1.3 Flows

Now, what is the situation of these results for flows? The dynamics of flows seems to be very related with the dynamics of diffeomorphisms. In some cases some results can be translated from one setting to the other, for instance by considering the suspension. And in even more cases, the dynamics of vector fields in dimension n looks like the one of diffeomorphisms in dimension $n-1$. For example, [D] proved that robustly transitive flows on 3-manifolds are Anosov flows, generalizing Mañé's result for surface diffeomorphisms. But in fact, there are several steps in the theory that seem to take longer time to be understood for flows than for the diffeomorphism setting, and several that are not yet understood! This is because in fact, for flows there are other difficulties that are particular from this setting. For example, there is a phenomenon which is really specific to vector fields: the existence of singularities (zero of the vector field) accumulated, in a robust way by regular recurrent orbits. This has being one of the main obstacles in reobtaining the results we have for diffeomorphisms in the flow setting. In fact when there are singularities robustly mixed with the regular recurrent orbits, some of the previously mentioned results may fail to be reobtained in the flow setting.

The first example with this behavior has been indicated by Lorenz in [Lo] under numerical evidences. Then [GuWi] constructs a C^1-open set of vector fields in a 3-manifold, having a topological transitive attractor containing periodic orbits (that are all hyperbolic) and one singularity. The examples in [GuWi] are known as the geometric Lorenz attractors.

The Lorenz attractor is also an example of a robustly non-hyperbolic whether flow, showing that the result in [H] is not true anymore for flows. In dimension 3 the difficulties introduced by the robust coexistence of singularities and periodic orbits is now almost fully understood. In particular, Morales, Pacifico and Pujals (see [MPP]) defined the notion of *singular hyperbolicity*, which requires some compatibility between the hyperbolicity of the singularity and the hyperbolicity of the regular orbits. They prove that, for C^1-generic star flows on 3-manifolds, every chain recurrence class is singular hyperbolic. It was conjectured in [GWZ] that the same result could hold without the generic assumption. However [BaMo] built a star flow on a 3-manifold having a chain recurrence class which is not singular hyperbolic, contradicting the conjecture. We exhibit a very simple such example in Section 5.

In higher dimension the theory of the relation between robust properties and hyperbolic structures for singular flows presents several difficulties. There are very few examples, illustrating what are the possibilities. Let us mention [BLY] which builds a flow having a robustly chain recurrent attractor containing saddles of different indices.

On a joint work with Christian Bonatti we propose a general way for extending the usual notions of hyperbolicity (hyperbolicity, partial hyperbolicity, volume hyperbolicity, etc..) which are well defined on compact invariant sets far from the singularities, to the case of regular recurrent orbits accumulating singularities. In particular, we will propose a notion of (multi)singular hyperbolicity which generalizes the ones already defined. We show the power of this notion by paying a special attention to star flows.

On the last section we use the notion of singular volume partial hyperbolicity to recover the results from [BDP] in an open and dense set of C^1-vector fields.

1.4 A star flow with singularities of different index

Recall that a a system is *star* if every periodic orbit of every C^1-close system is hyperbolic.

There are already many results on the hyperbolic structure of the star flows, in dimensions larger than 3. The notion of singular hyperbolicity defined by [MPP] in dimension 3 admits a straightforward generalization to higher dimensions: each chain recurrence class admits a dominated, partially hyperbolic splitting in two bundles, one being uniformly contracted (resp. expanded) and the other being sectionally area expanding (resp. sectionally area contracting). Far from the singular points, the singular hyperbolicity is equivalent to the hyperbolicity, as the direction spanned by the vector field is neither contracted nor expanded: the uniform area expansion means the uniform expansion of the normal directions.

If the chain recurrence set of a vector field X can be covered by filtrating sets U_i in which the maximal invariant set Λ_i is singular hyperbolic, then X is a star flow. Conversely, [GLW] and [GWZ] prove that this property characterizes the generic star flows on 4-manifolds. In [GSW] the authors prove the singular hyperbolicity of generic star flows in any dimensions assuming an extra property: if two singularities are in the same chain recurrence class then they must have the same *s-index* (dimension of the stable manifold). Indeed, the singular hyperbolicity implies directly this extra property.

However in Section 6 we build an example of a star flow in dimension 5 admitting singularities of different indices which belong robustly to the same chain recurrence class. Moreover, the chain class is robustly chain transitive

This example cannot satisfy the singular hyperbolicity used in [GSW].

Theorem 2. *Let M be the manifold $S^3 \times \mathbb{RP}^2$. There is a C^1-open set $\mathcal{U}$ of $\mathcal{X}^1(M)$ so that every $X \in \mathcal{U}$ is such that*

— *$X \in \mathcal{U}$ is a star flow.*

— *There is a chain class C with 2 singularities σ_1 and σ_2 in C, such that the stable manifold of σ_1 is 3 dimensional and the stable manifold of σ_2 is 2 dimensional (in fact, the singularities are robustly in the same class)*

It was shown by [GWZ] that a robustly transitive chain recurrence class of a star flow must have all the singularities of the same index, therefore, the example constructed above is not robustly transitive, however in Section **??** we prove that it is robustly chain transitive. The only other example of such a behavior known before can be found in [BCGP].

1.5 Hyperbolic structures for flows

Our example in 6 can be done in such a way that it does not admit any dominated splitting of the tangent space for the flow: therefore the hyperbolic structure we will define does not lie on the tangent bundle. By itself, this fact is not a surprise: many hyperbolic structures for flows are not expressed in terms of the differential of the flow, but on its transverse structure (called the *linear Poincaré flow*). For instance, there exists a robustly transitive flow without dominated splitting, but its linear Poincaré flow needs to carry a dominated volume partially hyperbolic splitting. However the linear Poincaré flow is only defined far from the singularities.

In [GLW], the authors define the notion of *extended linear Poincaré flow* defined on some sort of blow-up of the singularities. Our notion of *multisingular hyperbolicity* will be expressed as the hyperbolicity of a well chosen reparametrization of this extended linear Poincaré flow, over a well chosen extension of the chain recurrence set.

Theorem 6 proves that this multisingular hyperbolicity characterizes the star flows in any dimensions: the multi-singular hyperbolic flows are star flows, and an open and dense subset of the star flows consists of multi-singular hyperbolic flows. We notice that the example in Section 5 as well as the ones in [BaMo] are multisingular hyperbolic.

The multisingular hyperbolicity is a way of making compatible the hyperbolic structure of the regular orbits with the one of the singularities. The same idea holds for weaker (uniform) forms of hyperbolicity. We will show that the notions of partial hyperbolicity, volume partial hyperbolicity (...) can be adapted for singular flows by multiplying the extended linear Poincaré flow by some well chosen cocycles. This will define the corresponding *(multi)singular* structures.

1.5.1 The extended linear Poincaré flow

We deal with a vector field whose flow does not preserve any dominated splitting. As said before, it is natural to look for the hyperbolic structure on the normal bundle by considering the linear Poincaré flow (see the precise definition in Section 2.3).

However, as we deal with singular flows, the linear Poincaré flow is not defined everywhere: it is not defined on the singularities. A way proposed by [GLW] for bypassing this difficulty is the so called *extended linear Poincaré flow* (see the precise definition in Section **??**); we present it roughly below.

- The linear Poincaré flow is the natural linear cocycle over the flow, on the normal bundle to the flow. The dynamics on the fibers is the quotient of the derivative of the flow by the direction of the flow: this is possible as the direction of the flow is invariant.
- The vector field X provides an embedding of $M \setminus Sing(X)$ into the projective tangent bundle $\mathbb{P}M$: to a point x one associates the line directed by $X(x)$. Recall that one point L of $\mathbb{P}M$ corresponds to a line of the tangent space at a point of M. The flow ϕ^t of X induces (by the action of its derivative $D\phi^t$) a topological flow $\phi^t_{\mathbb{P}}$ on $\mathbb{P}M$ which extends the flow of X.
- The projective tangent bundle $\mathbb{P}M$ admits a natural bundle called the *normal bundle* $\mathcal{N}$: the fiber over $L \in \mathbb{P}M$ (corresponding to a line $L \subset T_xM$) is the quotient $\mathcal{N}_L = T_xM/L$. The derivative of the flow $D\phi^t$ of X passes to the quotient on the normal bundle $\mathcal{N}$ in a linear cocycle over $\phi^t_{\mathbb{P}}$, called the *extended linear Poincaré flow* and denoted by $\psi^t_{\mathcal{N}}$.

1.5.2 The extended maximal invariant set

The next difficulty is to define a compact part of $\mathbb{P}M$ in which we would like to define an hyperbolic structure. We are interested in the dynamics of X in a compact region U on M, that is, to describe the maximal invariant set $\Lambda(X, U)$ in U. An important property is that the maximal invariant set depends upper-semi continuously on the vector field X. This property is fundamental in the fact that "having a hyperbolic structure" is a robust property.

The natural first approach would be to consider the closure of the lifts of all points in $\Lambda(X, U)$. More precisely we define the *lifted maximal* invariant set as:

$$\Lambda_{\mathbb{P}}(X, U) = \{\, L \in \mathbb{P}M \text{ such that } L =< X(x) > \text{ for every } x \in \Lambda(X, U) \,\} \,.$$

How ever this set does not vary semi continuously with X (there is an example in chapter 3), therefore we needs to consider a compact part of $\mathbb{P}M$, as small as possible, such that

— it contains all the direction spanned by $X(x)$ for $x \in \Lambda(X, U) \setminus Sing(X)$,

— it varies upper semi-continuously with X

The smallest set with this conditions was defined in [GLW]. In this case the authors consider a neighborhood $\mathcal{U}$ of X

$$\widetilde{\Lambda} = \{\, L \in \mathbb{P}M \text{ such that } L =< Y(x) > \text{ for every } x \in \Lambda(Y, U) \text{ and for every } Y \in \mathcal{U} \,\} \ .$$

We call this set the *pre extended maximal invariant set*

But one of the main reasons that make hyperbolic structures so useful is that they allow us to detect robust behaviors just by looking at one flow. Therefore making a definition of hyperbolicity that relies on information of all the perturbations of your given flow, is not very satisfying. Therefore in chapter 3 we define precisely a set that:

— it contains all the direction spanned by $X(x)$ for $x \in \Lambda(X, U) \setminus Sing(X)$,

— it varies upper semi-continuously with X in the C^1 toppology

— it does not depend of a given neighborhood of X

. Any way we give a short description below

In Section 3.2 we define the notion of *central space of a singular point* $\sigma \in U$. Then we call *extended maximal invariant set* and we denote by $B(X, U) \subset \mathbb{P}M$ the set of all the lines L contained

— either in the central space of a singular point in $\bar{U}$

— or directed by the vector $X(x)$ at a regular point $x \in \Lambda(X, U) \setminus Sing(X)$.

Proposition 22 proves that $B(X, U)$ varies upper semi-continuously with the vector field X.

In particular, the existence of a dominated splitting $\mathcal{N}_L = E_L \oplus F_L$ of the normal bundle $\mathcal{N}$ over $B(X, U)$ is a robust property.

1.5.3 The usual singular hyperbolicity

The usual notion of *singular hyperbolicity* consists in a dominated splitting of the flow on the maximal invariant set $\Lambda(X, U)$ if for every $x \in \Lambda(X, U)$ we have that

$$T_x M = E^{ss} \oplus E^{cu} \,,$$

so that one bundle is uniformly contracted (E^{ss}) and the other expands area in any 2 dimensional subspace of E^{cu} we choose (or the same for the reverse flow).

One can reformulate it in terms of the extended linear Poincaré flow $\psi^t_{\mathcal{N}}$ as follows: the cocycle $\psi^t_{\mathcal{N}}$ admits a dominated splitting $E \oplus_{\prec} F$ over $B(X, U)$ so that

- — E is uniformly contracted by $\psi^t_{\mathcal{N}}$,
- — consider the *reparametrized linear Poincaré flow* obtained by multiplying the extended linear Poincaré flow $\psi^t_{\mathcal{N}}$, on the normal space $\mathcal{N}_L$, $L \in B(X, U)$, by the expansion of the derivative of the flow of X in the direction L.

  Then the reparametrized linear Poincaré flow expands uniformly the vectors in F.

The proof that this two notions are equivalent, is an immediate adaptation of the results in the last section in [GLW].

The singular hyperbolicity is not symmetric: one reparametrizes the extended linear Poincaré flow only on one bundle of the dominated splitting.

1.6 The extended maximal invariant set carries the same hyperbolic properties as the pre extended.

Depending on the information we have, it is sometimes more usefull to define hyperbolic structures over the extended maximal invariant set $B(X, U)$ (If we have no information of our surrounding systems) or over $\widetilde{\Lambda}$ (if the information we have is over the regular orbits but we have information on the surrounding vector fields). For this reason we show that there is no inconsistency in doing it sometimes one way and some times the other.

Remark 2. Consider the open and dense set of vector fields whose singular points are all hyperbolic. In this open set the singularities depend continuously on the field. Then for every singular point σ, the projective center space $\mathbb{P}^c_\sigma$ varies upper semi continuously, and in particular the dimension $dimE^c_\sigma$ varies upper semi-continuously. As it is a non-negative integer, it is locally minimal and locally constant on a open and dense subset.

We will say that such a singular point has *locally minimal center space*.

A chain recurrent class admits a basis of filtrating neighborhood. That is, for any chain recurrence class we can find a sequence of neighborhoods ordered by inclusion $U_{n+1} \subset U_n$, such that $C = \bigcap U_n$ We define

$$\widetilde{\Lambda(C)} = \bigcap_n \widetilde{\Lambda(X, U_n)} \text{ and } B(C) = \bigcap_n B(X, U_n).$$

These two sets are independent of the choice of the sequence U_n.

Definition 3. We say that a chain recurrence class C has a given singular hyperbolic structure if $\widetilde{\Lambda(C)}$ carries this singular hyperbolic structure.

We prove

Theorem 3. *[j.w.Christian Bonatti]Let X be a vector field on a closed manifold, whose singular points are all hyperbolic and with locally minimal center spaces. Then for every $\sigma \in SingX$, a dominated splitting of the extended linear Poincaré in $\widetilde{\Lambda(C(\sigma))}$ extends on $B(C(\sigma))$. Furthermore*

- *suppose that there is a bundle of the domination over $\widetilde{\Lambda}$ that is contracting (or expanding) uniformly or in volume for the extended linear Poincaré flow, then the same is true for the domination over $B(X,U)$.*
- *suppose that there is a bundle of the domination over $\widetilde{\Lambda}$ that is contracting (or expanding) uniformly or in volume, then the same is true for the domination over $B(X,U)$.*
- *suppose there is a subset $S_F \subset S$ so that the reparametrized cocycle $h_F^t \psi_{\mathcal{N}}^t$ is uniformly (in volume) contracted (expanded) in restriction to the a bundle of the domination over $\widetilde{\Lambda}$ where h_F denotes*

$$h_F = \Pi_{\sigma \in S_F} h_\sigma.$$

Then the same is true for that bundle in the domination over $B(X,U)$.

1.7 Rough presentation of the results on star flows

We want to exhibit a definition of hyperbolicity which allows two singularities σ_1, σ_2 of different indices to be robustly in the same chain recurrent class C. For this, with Christian Bonatti we propose the following notion, which we will show later on, that an open and dense set of star flows carry.

1.7.1 The multisingular hyperbolicity

In our situation, the extended linear Poincaré flow $\psi_{\mathcal{N}}^t$ admits a dominated splitting $E \oplus_{\prec} F$ over $B(X,U)$, and the singular set $Sing(X) \cap U$ is divided in two sets:

- the set S_E of singular points whose stable space has the same dimension as the bundle E,
- and the set S_F of singular points whose unstable space has the same dimension as the bundle F.

We want:

- to reparametrize the cocycle $\psi_{\mathcal{N}}^t$ in restriction to E_L by the expansion in the direction L if and only if the line L is based at a point close to S_E;
- to reparametrize the cocycle $\psi_{\mathcal{N}}^t$ in restriction to F_L by the expansion in the direction L if and only if the line L is based at a point close to S_F.

This leads to a difficulty: the reparametrizing function needs to satisfying a cocycle relation.

In Section 4.1, we prove that there is a cocycle $\{h_E^t\}_{t\in\mathbb{R}}$, called a *center-stable cocycle* so that:

- $h_E^t(L)$ and $\frac{1}{h_E^t}(L)$ are uniformly bounded (independently of t), if L is based on a point x so that x and $\phi^t(x)$ are out of a small neighborhood of S_E, where ϕ^t denotes the flow of X;
- $h_E^t(L)$ is in a bounded ratio with the expansion of ϕ^t in the direction L, if L is based at a point x so that x and $\phi^t(x)$ are out of a small neighborhood of S_F.
- h_E^t depends continuously on X.

Analogously we get the notion of *center-unstable cocycles* $\{h_F^t\}$ by exchanging the roles of S_E and S_F in the properties above.

We are now ready to define our notion of multisingular hyperbolicity.

Definition 4. Let X be a C^1-vector field on a closed manifold M. Let U be a compact set. We say that X is *multisingular hyperbolic in U* if:

- the extended linear Poincaré flow admits a dominated splitting $E \oplus F$ over the extended maximal invariant set $B(U, X)$.
- every singular point in U is hyperbolic.
- the set of singular points in U is the disjoint union $S_E \cup S_F$ where the stable space of the points in S_E has the same dimension as the bundle E and the unstable space of the points in S_F has the same dimension has the bundle F
- the reparametrized linear Poincaré flow $h_E^t\psi_{\mathcal{N}}^t$ is uniformly contracted on E, where h_E^t is a center-stable cocycle,
- the reparametrized linear Poincaré flow $h_F^t\psi_{\mathcal{N}}^t$ is uniformly expanded on F, where h_F^t is a center-unstable cocycle,

Using the upper semi-continuous dependence of the set $B(U, X)$ on X, and of the continuous dependence of the center-stable and center unstable cocycles h_E and h_F on X, one proves

Proposition 4. *The multisingular hyperbolicity of X in U is a C^1-robust property.*

Remark 5. If all the singular points in U have the same index, that is, if S_E or S_F is empty, then the multisingular hyperbolicity is the same notion as the singular hyperbolicity.

1.7.2 Star flows and multisingular hyperbolicity

We are now ready to state our results

Theorem 5 (j.w.Christian Bonatti)**.** *If X is multisingular hyperbolic in U then X is a star flow on U: more precisely, for any vector field Y C^1-close enough from X any periodic orbit contained in U is hyperbolic.*

This follows from the robustness of the multisingular hyperbolicity and the fact that it induces the usual hyperbolic structure on the periodic orbits.

We only get the converse for generic star flows:

Theorem 6 (j.w.Christian Bonatti)**.** *There is a C^1-open and dense subset $\mathcal{U}$ of $\mathcal{X}^{1*}(M)$ so that if $X \in \mathcal{U}$ is a star flow then the chain recurrent set $\mathcal{R}(X)$ is contained in the union of finitely many pairwise disjoint filtrating regions in which X is multisingular hyperbolic.*

The proof of Theorem 6 follows closely the proof in [GSW] that star flows with only singular points of the same index are singular hyperbolic.

Question 7. *Can we remove the generic assumption, at least in dimension 3, in Theorem 6? In other word, is it true that, given any star flow X (for instance on a 3-manifold) every chain recurrence class of X is multisingular hyperbolic?*

1.8 Rough presentation of our results on Robustly chain transitive classes

1.8.1 The singular volume partial hyperbolicity

In this section we take a closer look at weaker forms of hyperbolicity and their relation with the persistence of the dynamical properties.

Following the proofs in [BDP] we show the following

Proposition 8. *Let $\mathcal{U} \subset \mathcal{X}^1(M)$ be a C^1-open set such that, for every $X \in \mathcal{U}$ there is an open set U of M such that the maximal invariant set in U is an isolated chain recurrence class C. Then $\widetilde{\Lambda}$ has a uniform finest dominated splitting for the linear Poincaré flow:*

$$\mathcal{N}_L = \mathcal{N}_L^1 \oplus \cdots \oplus \mathcal{N}_L^n .$$

Each of this periodic orbits is volume partial hyperbolic for the tangent space.

We divide the singular set $Sing(X) \cap U$ in subsets sets:

— the set S_{Ec} of singular points whose escaping stable space has dimension smaller than $\mathcal{N}_L^1$,

- the set S_E of singular points whose escaping stable space has dimension bigger or equal than $\mathcal{N}_L^1$,
- the set S_{Fc} of singular points whose escaping unstable space has dimension smaller than $\mathcal{N}_L^n$,
- the set S_F of singular points whose escaping unstable space has dimension bigger or equal than $\mathcal{N}_L^n$.

We want:

- to reparametrize the cocycle $\psi_{\mathcal{N}}^t$ in restriction to $\mathcal{N}_L^1$ by the expansion in the direction L if and only if the line L is based at a point close to S_{Ec};
- to reparametrize the cocycle $\psi_{\mathcal{N}}^t$ in restriction to $\mathcal{N}_L^n$ by the expansion in the direction L if and only if the line L is based at a point close to S_{Fc}.

For this we use again the cocycle *center-stable cocycle* $\{h_{Ec}^t\}_{t\in\mathbb{R}}$,so that:

- $h_{Ec}^t(L)$ and $\frac{1}{h_{Ec}^t}(L)$ are uniformly bounded (independently of t), if L is based on a point x so that x and $\phi^t(x)$ are out of a small neighborhood of S_{Ec}, where ϕ^t denotes the flow of X;
- $h_{Ec}^t(L)$ is in a bounded ratio with the expansion of ϕ^t in the direction L, if L is based at a point x so that x and $\phi^t(x)$ are out of a small neighborhood of S_E.
- h_{Ec}^t depends continuously on X.

Analogously we get the notion of *center-unstable cocycles* $\{h_{Fc}^t\}$ by exchanging the roles of S_{Ec} and S_{Fc} in the properties above.

Now similarly to the multisingular hyperbolicity case, we define the singular volume partial hyperbolicity.

Definition 6. Let X be a C^1-vector field on a closed manifold M. Let U be a compact set. We say that X is *singular volume partial hyperbolic in U* if:

- the extended linear Poincaré flow admits a finest dominated splitting $\mathcal{N}_L = \mathcal{N}_L^1 \oplus \cdots \oplus \mathcal{N}_L^n$. over the pre extended maximal invariant set $\widetilde{\Lambda}$.
- the set of singular points in U is the union of

$$S_{Ec} \cup S_{Fc} \cup S_E \cup S_F \,,$$

  defined above.
- the reparametrized linear Poincaré flow $h_{Ec}^t\psi_{\mathcal{N}}^t$ contracts volume on $\mathcal{N}_L^1$, where h_{Ec}^t is a center-stable cocycle,
- the reparametrized linear Poincaré flow $h_{Fc}^t\psi_{\mathcal{N}}^t$ contracts volume on $\mathcal{N}_L^n$, where h_{Fc}^t is a center-unstable cocycle,

1.8.2 Robustly chain transitive singular sets

We now state the main result of this section:

Theorem 9. *There is a C^1-open and dense set $\mathcal{G} \subset \mathcal{X}^1(M)$, such that for any $\mathcal{U} \subset \mathcal{G}$ a C^1-open set such that if $X \in \mathcal{U}$ and there is an open set U of M in which the maximal invariant set is an isolated chain recurrence class C, then X is singular volume partial hyperbolic in U.*

Chapter 2

Preliminaries

2.1 Linear cocycle

Let $\phi = \{\phi^t\}_{t\in\mathbb{R}}$ be a topological flow on a compact metric space K. A *linear cocycle* (A, E) *over* (K, ϕ) is a continuous map $A^t\colon E \times \mathbb{R} \to E$ defined by

$$A^t(x, v) = (\phi^t(x), A_t(x)v)\,,$$

where

— $\pi\colon E \to K$ is a d dimensional linear bundle over K;

— $A_t : (x, t) \in K \times \mathbb{R} \mapsto GL(E_x, E_{\phi^t(x)})$ is a continuous map that satisfies the *cocycle relation* :

$$A_{t+s}(x) = A_t(\phi^s(x))A_s(x), \quad \text{for any } x \in K \text{ and } t, s \in \mathbb{R}$$

Note that $\mathcal{A} = \{A^t\}_{t\in\mathbb{R}}$ is a flow on the space E which projects on ϕ^t.

$$\begin{array}{ccc} E & \xrightarrow{A^t} & E \\ \downarrow & & \downarrow \\ K & \xrightarrow{\phi^t} & K \end{array}$$

If $\Lambda \subset K$ is a ϕ-invariant subset, then $\pi^{-1}(\Lambda) \subset E$ is $\mathcal{A}$-invariant, and we call *the restriction of $\mathcal{A}$ to Λ* the restriction of $\{A^t\}$ to $\pi^{-1}(\Lambda)$.

2.1.1 Hyperbolicity, dominated splitting on linear cocycles

Definition 7. Let ϕ be a topological flow on a compact metric space Λ. We consider a vector bundle $\pi\colon E \to \Lambda$ and a linear cocycle $\mathcal{A} = \{A^t\}$ over (Λ, ϕ).

We say that $\mathcal{A}$ admits a *dominated splitting over* Λ if

— there exists a splitting $E = E^1 \oplus \cdots \oplus E^k$ over λ into k subbundles

— The dimension of the sub bundles is constant, i.e. $dim(E^i_x) = dim(E^i_y)$ for all $x, y \in \Lambda$ and $i \in \{1 \ldots k\}$,

— The splitting is invariant, i.e. $A^t(x)(E^i_x) = E^i_{\phi^t(x)}$ for all $i \in \{1 \ldots k\}$,

— there exists a $t > 0$ such that for every $x \in \Lambda$ and any pair of non vanishing vectors $v \in E^i_x$ and $u \in E^j_x$, $i < j$ one has

$$\frac{\| A^t(u) \|}{\| u \|} \leq \frac{1}{2} \frac{\| A^t(v) \|}{\| v \|} \tag{2.1}$$

We denote $E^1 \oplus_{\prec} \cdots \oplus_{\prec} E^k$.the splitting is *t-dominated*.

A classical result (see for instance [BDV, Appendix B]) asserts that the bundles of a dominated splitting are always continuous. A given cocycle may admit several dominated splittings. However, the dominated splitting is unique if one prescribes the dimensions $dim(E^i)$.

We can consider metric spaces K os invariant subspaces Λ of K that are not compact. In this case we would ask for the norm of $\mathcal{A}$ to be bounded and Note that the dominated splitting defined as above is uniform with respect to the point. This is particularly important when we consider a dominated splitting over a set that is not compact.

Associated to the dominated splitting we define a family of cone fields C^{iu}_a around each space $E^i \oplus \cdots \oplus E^k$ as follows. Let us write the vectors $v \in E$ as $v = (v_1, v_2)$ with $v_1 \in E^1 \oplus \cdots \oplus E^{i-1}$ and $v_2 \in E^i \oplus \cdots \oplus E^k$. Then the cone field C^{iu}_a is the set

$$C^{iu}_a = \{ v = (v_1, v_2) \text{ such that } \| v_1 \| < a \| v_2 \| \} .$$

These are called the *family of unstable conefields* and the domination gives us that they are strictly invariant for times larger than t: i.e. the cone C^{iu}_a at $T_x M$ is taken by A^t to the interior of the cone C^{iu}_a at $T_{\phi^t x} M$.

Analogously we define the *stable family of conefields* C^{is}_a around $E^1 \oplus \cdots \oplus E^i$ and the domination gives us that they are strictly invariant for negative times smaller than $-t$.

One says that one of the bundle E^i is *(uniformly) contracting* (resp. *expanding*) if there is $t > 0$ so that for every $x \in \Lambda$ and every non vanishing vector $u \in E^i_x$ one has $\frac{\| A^t(u) \|}{\| u \|} < \frac{1}{2}$ (resp. $\frac{\| A^t(u) \|}{\| u \|} < \frac{1}{2}$). In both cases one says that E^i is *hyperbolic*.

Notice that is E^j is contracting (resp. expanding) then the same holds for any E^i, $i < j$ (reps. $j < i$) as a consequence of the domination.

Definition 8. We say that the linear cocycle $\mathcal{A}$ is *hyperbolic over* Λ if there is a dominated splitting $E = E^s \oplus_{\prec} E^u$ over Λ into 2 hyperbolic sub bundles so that E^s is uniformly contracting and E^u is uniformly expanding.

One says that E^s is the *stable bundle*, and E^u is the *unstable bundle*.

The existence of a dominated splitting or of a hyperbolic splitting is an open property in the following sense:

Proposition 10. *Let K be a compact metric space, $\pi\colon E \to K$ be a d-dimensional vector bundle, and $\mathcal{A}$ be a linear cocycle over K. Let Λ_0 be a ϕ-invariant compact set. Assume that the restriction of $\mathcal{A}$ to Λ_0 admits a dominated splitting $E^1 \oplus_{\prec t} \cdots \oplus_{\prec t} E^k$, for some $t > 0$.*

Then there is a compact neighborhood $U \subset K$ of Λ_0 with the following property. Let $\Lambda = \bigcap_{t\in\mathbb{R}} \phi^t(U)$ be the maximal invariant set of ϕ in U. Then the dominated splitting admits a unique extension as a $2t$-dominated splitting over Λ. Furthermore if one of the sub bundles E^i is hyperbolic over Λ_0 it is still hyperbolic over Λ.

As a consequence, if $\mathcal{A}$ is hyperbolic over Λ_0 then (up to shrink U if necessary) it is also hyperbolic over Λ.

2.1.2 Robustness of hyperbolic structures

The aim of this section is to explain that Proposition 10 can be seen as a robustness property.

Let M be a manifold and ϕ_n be a sequence of flows tending to ϕ_0 as $n \to +\infty$, in the C^0-topology on compact subsets: for any compact set $K \subset M$ and any $T > 0$, the restriction of ϕ_n^t to K, $t \in [-T, T]$, tends uniformly (in $x \in K$ and $t \in [-T, T]$) to ϕ_0^t.

Let Λ_n be compact ϕ_n-invariant subsets of M, and assume that the upper limit of the Λ_n for the Hausdorff topology is contained in Λ_0: more precisely, any neighborhood of Λ_0 contains all but finitely many of the Λ_n. Let us present another way to see this property:

Consider the subset $\mathcal{I} = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}$ endowed with the induced topology. Consider $M_\infty = M \times \mathcal{I}$. Let Λ_∞ denote

$$\Lambda_\infty = \Lambda_0 \times \{0\} \cup \bigcup_{n>0} \Lambda_n \times \{\frac{1}{n}\} \subset M_\infty.$$

With this notation, the upper limit of the Λ_n is contained in Λ_0 if and only if Λ_∞ is a compact subset.

Let $\pi\colon E \to M$ be a vector bundle. We denote $E_\infty = E \times \mathcal{I}$ the vector bundle $\pi_\infty\colon E \times \mathcal{I} \to M \times \mathcal{I}$. We denote by $E_\infty|_{\Lambda_\infty}$ the restriction of E_∞ on the compact subset Λ_∞.

Assume now that $\mathcal{A}_n$ are linear cocycles over the restriction of E to Λ_n. We denote by $\mathcal{A}_\infty$

the map defined on the restriction $E_\infty|_{\Lambda_\infty}$ by:

$$\begin{aligned} A^t_\infty(x,0) &= (A^t_0(x),0), \text{ for } (x,0)\in\Lambda_0\times\{0\} \text{ and} \\ A^t_\infty(x,\frac{1}{n}) &= (A^t_n(x),\frac{1}{n}), \text{ for } (x,0)\in\Lambda_n\times\{\frac{1}{n}\}. \end{aligned}$$

Definition 9. With the notation above, we say that the family of cocycles $\mathcal{A}_n$ tends to $\mathcal{A}_0$ as $n\to\infty$ if the map $\mathcal{A}_\infty$ is continuous, and therefore is a linear cocycle.

As a consequence of Proposition 10 we get:

Corollary 11. *Let $\pi\colon E\to M$ be a linear cocycle over a manifold M and let ϕ_n be a sequence of flows on M converging to ϕ_0 as $n\to\infty$. Let Λ_n be a sequence of ϕ_n-invariant compact subsets so that the upper limit of the Λ_n, as $n\to\infty$, is contained in Λ_0.*

Let $\mathcal{A}_n$ be a sequence of linear cocycles over ϕ_n defined on the restriction of E to Λ_n. Assume that $\mathcal{A}_n$ tend to $\mathcal{A}_0$ as $n\to\infty$.

Assume that $\mathcal{A}_0$ admits a dominated splitting $E=E^1\oplus_\prec\cdots\oplus_\prec E^k$ over Λ_0. Then, for any n large enough, $\mathcal{A}_n$ admits a dominated splitting with the same number of sub-bundles and the same dimensions of the sub-bundles. Furthermore, if E^i was hyperbolic (contracting or expanding) over Λ_0 it is still hyperbolic (contracting or expanding, respectively) for $\mathcal{A}_n$ over Λ_n.

The proof just consist in applying Proposition 10 to a neighborhood of $\Lambda_0\times\{0\}$ in Λ_∞.

Corollary 12. *Let $\pi\colon E\to M$ be a linear cocycle over a manifold M and let ϕ_n be a sequence of flows on M converging to ϕ_0 as $n\to\infty$. Let Λ_n be a sequence of ϕ_n-invariant compact subsets so that the upper limit of the Λ_n, as $n\to\infty$, contains Λ_0.*

Let $\mathcal{A}_n$ be a sequence of linear cocycles over ϕ_n defined on the restriction of E to Λ_n. Assume that $\mathcal{A}_n$ tend to $\mathcal{A}_0$ as $n\to\infty$.

Assume that $\mathcal{A}_n$ admits a dominated splitting $E=E^1\oplus_\prec\cdots\oplus_\prec E^k$ over Λ_n. Then, $\mathcal{A}_0$ admits a dominated splitting over Λ_0, with the same number of sub-bundles and the same dimensions of the sub-bundles.

2.2 Linear Poincaré flow

Let X be a C^1 vector field on a compact manifold M. We denote by ϕ^t the flow of X.

Definition 10. The *normal bundle* of X is the vector bundle N_X over $M\setminus Sing(X)$ defined as follows: the fiber $N_X(x)$ of $x\in M\setminus Sing(X)$ is the quotient space of TxM by the vector line $\mathbb{R}.X(x)$.

Note that, if M is endowed with a Riemannian metric, then $N_X(x)$ is canonically identified with the orthogonal space of $X(x)$:

$$N_X = \{(x, v) \in TM, v \perp X(x)\}$$

Consider $x \in M \setminus Sing(M)$ and $t \in \mathbb{R}$. Thus $D\phi^t(x) : T_xM \to T_{\phi^t(x)}M$ is a linear automorphism mapping $X(x)$ onto $X(\phi^t(x))$. Therefore $D\phi^t(x)$ passes to the quotient as a linear automorphism $\psi^t(x) \colon N_X(x) \to N_X(\phi^t(x))$:

$$\begin{array}{ccc} T_xM & \xrightarrow{D\phi^t} & T_{\phi^t(x)}M \\ \downarrow & & \downarrow \\ N_X(x) & \xrightarrow{\psi^t} & N_X(\phi^t(x)) \end{array}$$

where the vertical arrow are the canonical projection of the tangent space to the normal space parallel to X.

Proposition 13. *Let X be a C^1 vector field on a manifold and Λ be a compact invariant set of X. Assume that Λ does not contain any singularity of X. Then Λ is hyperbolic if and only if the linear Poincaré flow over Λ is hyperbolic.*

Notice that the notion of dominated splitting for non-singular flows is sometimes better expressed in term of linear Poincaré flow: for instance, if one consider the suspension of robustly transitive diffeomorphism without partially hyperbolic splitting (as built in [**?**]) one gets a robustly transitive vector field X whose flow $\{\phi^t\}$ does not admit any dominated splitting.

2.3 Extended linear Poincaré flow

We are dealing with singular flows and the linear Poincaré flow is not defined on the singularity of the vector field X. However we can include the linear Poincaré flow in a flow, called *extended linear Poincaré flow* defined in [GLW], on a larger set, and for which the singularities of X do not play a specific role.

This flow will be a linear cocycle define on some linear bundle over a manifold, that we define now.

Definition 11. Let M be a manifold of dimension d.

— We call *the projective tangent bundle of M*, and denote by $\Pi_{\mathbb{P}} \colon \mathbb{P}M \to M$, the fiber bundle whose fiber $\mathbb{P}_x$ is the projective space of the tangent space T_xM: in other words, a point $L_x \in \mathbb{P}_x$ is a 1-dimensional vector subspace of T_xM.

— We call *the tautological bundle of* $\mathbb{P}M$, and we denote by $\Pi_{\mathcal{T}}\colon \mathcal{T}M \to \mathbb{P}M$, the 1-dimensional vector bundle over $\mathbb{P}M$ whose fiber $\mathcal{T}_L$, $L \in \mathbb{P}M$, is the the line L itself.

— We call *normal bundle of* $\mathbb{P}M$ and we denote by $\Pi_{\mathcal{N}}\colon \mathcal{N}M \to \mathbb{P}M$, the $d-1$-dimensional vector bundle over $\mathbb{P}M$ whose fiber $\mathcal{N}_L$ over $L \in \mathbb{P}_x$ is the quotient space T_xM/L.
If we endow M with riemannian metric, then $\mathcal{N}_L$ is identified with the orthogonal hyperplane of L in T_xM.

Let X be a C^r vector field on a compact manifold M, and ϕ^t its flow. The natural actions of the derivative of ϕ^t on $\mathbb{P}M$ and $\mathcal{N}M$ define C^{r-1} flows on these manifolds. More precisely, for any $t \in \mathbb{R}$,

— We denote by $\phi^t_{\mathbb{P}}\colon \mathbb{P}M \to \mathbb{P}M$ the C^{r-1} diffeomorphism defined by

$$\phi^t_{\mathbb{P}}(L_x) = D\phi^t(L_x) \in \mathbb{P}_{\phi^t(x)}.$$

— We denote by $\psi^t_{\mathcal{N}}\colon \mathcal{N}M \to \mathcal{N}M$ the C^{r-1} diffeomorphism whose restriction to a fiber $\mathcal{N}_L$, $L \in \mathbb{P}_x$, is the linear automorphisms onto $\mathcal{N}_{\phi^t_{\mathbb{P}}(L)}$ defined as follows: $D\phi^t(x)$ is a linear automorphism from T_xM to $T_{\phi^t(x)}M$, which maps the line $\mathcal{T}_L \subset T_xM$ onto the line $\mathcal{T}_{phi^t_{\mathbb{P}}(L)}$. Therefore it passe to the quotient in the announced linear automorphism.

$$\begin{array}{ccc}
T_xM & \xrightarrow{D\phi^t} & T_{\phi^t(x)}M \\
\downarrow & & \downarrow \\
\mathcal{N}_L & \xrightarrow{\psi^t_{\mathcal{N}}} & \mathcal{N}_{\phi^t_{\mathbb{P}}(L)}
\end{array}$$

Note that $\phi^t_{\mathbb{P}}$, $t \in \mathbb{R}$ defines a flow on $\mathbb{P}_M$ which is a co-cycle over ϕ^t whose action on the fibers is by projective maps.

The one-parameter family $\psi^t_{\mathcal{N}}$ defines a flow on $\mathcal{N}M$, which is a linear co-cycle over $\phi^t_{\mathbb{P}}$. We call $\psi^t_{\mathcal{N}}$ the *extended linear Poncaré flow*. We can summarize by the following diagrams:

$$\begin{array}{ccc}
\mathcal{N}M & \xrightarrow{\psi^t_{\mathcal{N}}} & \mathcal{N}M \\
\downarrow & & \downarrow \\
\mathbb{P}M & \xrightarrow{\phi^t_{\mathbb{P}}} & \mathbb{P}M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi^t} & M
\end{array}$$

Remark 12. The extended linear Poincaré flow is really an extension of the linear Poincaré flow defined in the previous section; more precisely:

Let $S_X\colon M \setminus Sing(X) \to \mathbb{P}M$ be the section of the projective bundle defined as $S_X(x)$ is the line $\langle X(x)\rangle \in \mathbb{P}_x$ generated by $X(x)$. Then $N_X(x) = \mathcal{N}_{S_X(x)}$ and the linear automorphisms $\psi^t\colon N_X(x) \to N_X(\phi^t(x))$ and $\psi^t_{\mathcal{N}}\colon \mathcal{N}_{S_X(x)} \to \mathcal{N}_{S_X(\phi^t(x))}$

2.4 Controlling Stable and unstable Manifolds

2.4.1 Pliss lemma and controlling invariant manifolds near singularities.

We now present some results that allow us a better control of the size of the invariant manifolds near singularities. We need for this the definition of $(\eta, T, E)^*$ contracting orbit arcs.

Definition 13. Given ϕ_t a flow induced by $X \in \mathcal{X}^1(M)$, Λ a compact invariant set of ϕ_t, and $E \subset \|\, \Lambda - Sing(X) \,\|$ an invariant bundle of the linear Poincaré flow ψ_t. For $\eta > 0$ and $T > 0$, $x \in \Lambda - Sing(X)$ is called $(\eta, T, E)^*$ *contracting* if for any $n \in \mathbb{N}$,

$$\prod_{i=0}^{n-1} \left\| \psi^*_{T|E(\phi_{iT}(x))} \right\| \le e^{-n\eta}.$$

Similarly $x \in \Lambda - Sing(X)$ is called $(\eta, T, F)^*$ expanding if it is $(\eta, T, F)^*$ contracting for $-X$.

To find the $(\eta, T, E)^*$ contracting orbit arcs, one needs the classical result due to V.Pliss:

Lemma 14. *[P2](Pliss lemma) Given a number A. Let $\{a_1, \cdots, a_n\}$ be a sequence of numbers which are bounded from above by A. Assume that there exists a number $\xi < A$ such that $\sum_{i=1}^{n} a_i \ge n \cdot \xi$, then for any $\xi' < \xi$, there exist l integers $1 \le t_1 < \cdots < t_l \le n$ such that*

$$\frac{1}{t_j - k} \sum_{i=k+1}^{t_j} a_i \ge \xi', \text{for any } j = 1, \cdots, l \text{ and any integer } k = 0, \cdots t_j - 1.$$

Moreover, one has the estimate $\frac{l}{n} \ge \frac{\xi - \xi'}{A - \xi'}$.

Under certain situation, a point can be $(\eta, T, E)^*$ contracting and $(\eta, T, F)^*$ expanding at the same time:

Lemma 15. *Let ϕ_t be a flow induced by a C^1 vector field. Given a periodic point p. Assume that there exists an ψ_t invariant splitting $\mathcal{N}_{Orb(p)} = E \oplus F$, where ψ_t is the linear Poincare flow for ϕ_t. Assume, in addition, that there exist an integer m, a positive number T number and a number $\eta < 0$ such that for any point $x \in Orb(p)$, one has the followings:*

—
$$\frac{\| \psi_T|_{E(x)} \|}{\min(\psi_T|_{F(x)})} < \exp(2\eta);$$

—
$$\prod_{i=0}^{[m\pi(p)/T]-1} \| \psi_T|_{E(\phi_{iT}(x))} \| < \exp(\eta)$$

—
$$\prod_{i=0}^{[m\pi(p)/T]-1} \min(\psi_T|_{F(\phi_{iT}(x))}) > \exp(-\eta)$$

For any $\eta' \in (0, \eta)$, there exists a point $p' \in Orb(p)$ such that p' is both $(\eta, T, E)^*$ contracting and $(\eta, T, F)^*$ expanding.

We call the point p' as (η', T) bi-pliss point *or* bi-pliss point *for simplicity.*

Definition 14. Let M be a compact Riemannian manifold with a metric d. Let A be a submanifold of M of dimension i. We say that A has inner diameter bigger than k at x if the ball of center x and radius k for M intersects A in a i-dimensional of center x and radius k for the restriction of d to A.

Theorem 16. *[L1] Let $X \in \mathcal{X}^1(M)$ and Λ be a compact invariant set of ϕ_t associated to X. Given $\eta > 0$, $T > 0$ assume that $\| \Lambda - Sing(X) \| = E \oplus F$ is an (η, T)-dominated splitting with respect to the linear Poincaré flow. Then there is $\delta > 0$ such that if x is $(\eta, T, E)^*$ contracting, then the inner diameter of the stable manifold of x at x, is bigger than $\delta \| X(x) \|$.*

Remark 15. From **??** and 14 we have that for the periodic orbits of the star flows we can always find $(\eta, T, E)^*$ contracting and $(\eta, T, F)^*$ expanding points in any periodic orbit. Moreover, there are points in every periodic orbits that are both $(\eta, T, E)^*$ contracting and $(\eta, T, F)^*$ expanding.

2.5 Generic properties

We say that a vector field is Kupka-Smale if the following two properties hold

— All periodic and singular orbits are hyperbolic.

— The intersections of stable and unstable manifolds of closed hyperbolic orbits are transversal

A famous theorem by Kupka and Smale show that this conditions are generic

Lemma 17 (Connecting lemma)**.** *[BC]. Given ϕ_t induced by a Kupka-Smale vector field $X \in \mathcal{X}^(M)$. For any C^1 neighborhood U of X and $x, y \in M$ if y is attainable from x, then there exists*

$Y \in U$ and $t > 0$ such that $\phi_t^Y(x) = y$. Moreover, for every $k \geq 1$, let $\{x_{i,k}, t_{i,k}\}_{i=0}^{n_k}$ be a $(1/k, T)$-pseudo orbit from x to y and denote by

$$\Delta_k = \bigcup_{i=0}^{n_k-1} \phi_{[0,t_{i,k}]}(x_{i,k}).$$

Let Δ be the upper Hausdorff limit of Δ_k. Then for any neighborhood U of Δ, there exists $Y \in U$ with $Y = X$ on $M - U$ and $t > 0$ such that $\phi_t^Y(x) = y$.

Remark 16. From the proof of connecting lemma for pseudo orbit,one can obtain the following stronger statement: for any neighborhood U of Δ, and for any finitely many (hyperbolic) critical elements $c_i,\ i = 1, \ldots, j$, there exists a neighborhood V_i of c_i $\forall i$ and $Y \in U$ with $Y = X$ on $(M - U) \cup (\cup_{i=1}^{j} V_i)$ and $t > 0$ such that $\phi_t^Y(x) = y$.

The following result is a consequence of Connecting Lemma for pseudo-orbits made by S. Crovisier and C. Bonatti in [BC].

Theorem 18. *[C] There exists a $G_{approx} \subset \mathcal{X}^1(M)$ a generic set such that for every $X \in G_{approx}$ and for every C a chain recurrence class there exists a sequence of periodic orbits γ_n which converges to C in the Hausdorff topology.*

2.6 Hyperbolicity measures and periodic orbits

The following theorem by Mañe was first introduced in [Ma] and it was used in [Ma2] to prove the stability conjecture. The idea behind this theorem is that the lack of hyperbolicity in a set can be detected by the clack of hyperbolicity in a periodic orbit of a C^1 close system.

Definition 17. Let f be a diffeomorphism of a compact manifold M with a Riemmanian metric d . A point x is *well closable* if for every ϵ there are diffeomorphisms g, that are $\epsilon - C^1$ close to f and periodic points y for g with period T_y, such that

$$d(f^i(x), g^i(y)) < \epsilon \text{ for all } 0 \leq i \leq T_y\,.$$

We note the set of well closable points of f as $\mathcal{W}(f)$

Theorem (*Ergodic closing lemma*)**.** *[Ma] Let f be a diffeomorphism and μ an $f-$invariant probability measure, then almost every point is well closable. That is*

$$\mu\mathcal{W}(f) = 1\,.$$

The version of this theorem for flows is almost the same with one exception: the well closable points might be closed y a singularity.

Definition 18. Let X be a vector field of a compact manifold M with a Riemannian metric d,and ϕ its associated flow. A point x is *well closable* if for every ϵ there are vector fields Y, that are $\epsilon - C^1$ close to X and critical elements (closed orbits) y for Y with period T_y, such that

$$d(\phi_t^X(x), \phi_t^Y(y)) < \epsilon \text{ for all } 0 \leq t \leq T_y\,.$$

We note the set of well closable points of X as $\mathcal{W}(X)$

Theorem (*Ergodic closing lemma for flows*)**.** *[Ma][W] Let X be a vector field of a compact manifold M with a Riemannian metric d. For every $T > 0$ and μ a ϕ_T-invariant probability measure, almost every point is well closable. That is*

$$\mu\mathcal{W}(f) = 1\,.$$

The difference between one version and the other is not trivial. In order to make good use of the ergodic closing lemma for flows, one has to understand the behavior, not only of the periodic orbits of the surrounding systems, but also the behavior of the singularities.

Chapter 3

The extended maximal invariant set

The goal of this section is to define a subset in $\mathbb{P}M$ that varies upper semi continuously in the C^1 topology with the flow X and that contains all the directions spanned by the vector field over points in a maximal invariant set. We want this set to be as small as possible. In [GLW] the authors propose a set with this property. We introduce this notion and we propose another one that is not depending on the perturbations of the vector field.

3.1 Strong stable, strong unstable and center spaces associated to a hyperbolic singularity.

Let X be a vector field and $\sigma \in Sing(X)$ be a hyperbolic singular point of X. Let $\lambda_k^s \dots \lambda_2^s < \lambda_1^s < 0 < \lambda_1^u < \lambda_2^u \dots \lambda_l^u$ be the Lyapunov exponents of ϕ_t at σ and let $E_k^s \oplus_< \cdots E_2^s \oplus_< E_1^s \oplus_< E_1^u \oplus_< E_2^u \oplus_< \cdots \oplus_< E_l^u$ be the corresponding (finest) dominated splitting over σ.

A subspace F of $T_\sigma M$ is called a *center subspace* if it is of one of the possible form below:

— Either $F = E_i^s \oplus_< \cdots E_2^s \oplus_< E_1^s$

— Or $F = E_1^u \oplus_< E_2^u \oplus_< \cdots \oplus_< E_j^u$

— Or else $F = E_i^s \oplus_< \cdots E_2^s \oplus_< E_1^s \oplus_< E_1^u \oplus_< E_2^u \oplus_< \cdots \oplus_< E_j^u$

A subspace of $T_\sigma M$ is called a *strong stable space*, and we denote it $E_i^{ss}(\sigma)$, if there in $i \in \{1, \dots, k\}$ such that:

$$E_i^{ss}(\sigma) = E_k^s \oplus_< \cdots E_{i+1}^s \oplus_< E_i^s$$

A classical result from hyperbolic dynamics asserts that for any i there is a unique injectively immersed manifold $W_i^{ss}(\sigma)$, called a *strong stable manifold* tangent at $E_i^{ss}(\sigma)$ and invariant by the flow of X.

We define analogously the *strong unstable spaces* $E_j^{uu}(\sigma)$ and the *strong unstable manifolds* $W_j^{uu}(\sigma)$ for $j = 1, \dots, l$.

3.2 The lifted maximal invariant set and the singular points

Let U be a compact region, and X a vector field. Let $\sigma \in Sing(X) \cap U$ be a hyperbolic singularity of X contained in U and Λ the maximal invariant set in U.

We define de *lifted maximal invariant set*, that is:

$$\Lambda_{\mathbb{P}}(X,U) = \overline{\{ < X(x) > \in \mathbb{P}M \text{ such that } x \in \Lambda \}}.$$

The lifted maximal invariant set does not vary semi-continuously with X.

In fact we can consider a flow with two hyperbolic singularities σ_1 and σ_2 in $\mathbb{R}^3$, such that they have a strongly contracting space, a weakly contracting space and an unstable space. Suppose as well that there is a cycle between σ_1 and σ_2 connecting the strong stable and the unstable manifolds of σ_1 and σ_2 respectively , and let $\Lambda(X,U)$ be the cycle and the singularities. Then the directions tangent to the weak unstable spaces are not in $\Lambda_{\mathbb{P}}(X,U)$ but a small perturbation un the stable manifold of σ_1 can make the orbit corresponding to the branch in the cycle of the unstable manifold of σ_2, to approach σ_1 almost tangent to the weak stable value. Therefore, for this perturbation of our vector field the weak stable space of σ_1 is in $\Lambda_{\mathbb{P}}(X,U)$.

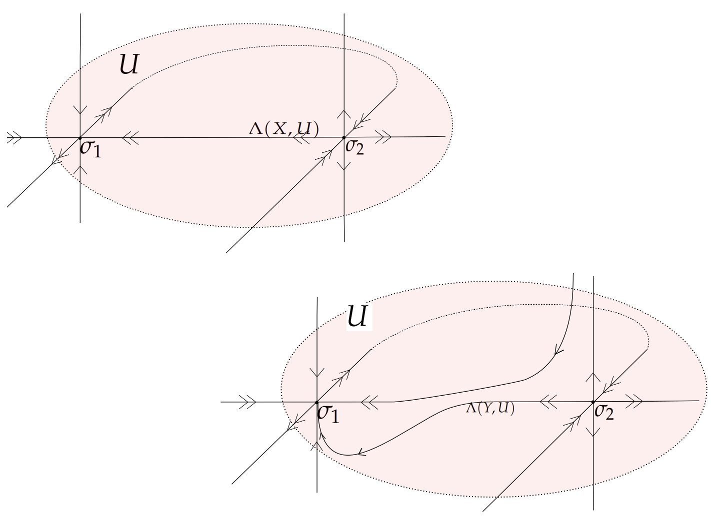


Figure 3.1 – A perturbation of X to a vectorfield Y such that there is a small neighborhood of $\Lambda_{\mathbb{P}}(X,U)$ nos containing $\Lambda(Y,U)$

Our aim is to add to the lifted maximal invariant set $\Lambda_{\mathbb{P}}(X, U)$, some set over the singular points (that do not depend on the perturbations of X)in order to recover some upper semi-continuity properties.

We define the *escaping stable space* $E^{ss}_{\sigma,U}$ as the biggest strong stable space $E^{ss}_j(\sigma)$ such that the corresponding strong stable manifold $W^{ss}_j(\sigma)$ is *escaping*, that is:

$$\Lambda(X, U) \cap W^{ss}_j(\sigma) = \{\sigma\}.$$

We define the *escaping unstable space* analogously.

We define the *central space* $E^c_{\sigma,U}$ *of* σ *in* U the center space such that

$$T_\sigma M = E^{ss}_{\sigma,U} \oplus E^c_{\sigma,U} \oplus E^{uu}_{\sigma,U}$$

We denote by $\mathbb{P}^i_{\sigma,U}$ the projective space of $E^i(\sigma, U)$ where $i \in \{ ss, uu, c \}$.

Lemma 19. *Let U be a compact region and X a vector field whose singular points are hyperbolic. Then, for any $\sigma \in Sing(X) \cap U$, one has :*

$$\Lambda_{\mathbb{P}}(X, U) \cap \mathbb{P}^{ss}_{\sigma,U} = \Lambda_{\mathbb{P}}(X, U) \cap \mathbb{P}^{uu}_{\sigma,U} = \emptyset.$$

Proof. Suppose (arguing by contradiction) that $L \in \Lambda_{\mathbb{P}}(X, U) \cap \mathbb{P}^{ss}_{\sigma,U}$. There exist a sequence $x_n \in \Lambda_{X,U} \setminus Sing(X)$ converging to σ, such that L_{x_n} converge to L, where L_{x_n} is the line $\mathbb{R}X(x_n) \in \mathbb{P}_{x_n}$.

We fix a small neighborhood V of σ endowed with local coordinates so that the vector field is very close to its linear part in these coordinates: in particular, there is a small cone $C^{ss} \subset V$ around $W^{ss}_{\sigma,U}$ so that the complement of this cone is strictly invariant in the following sense: the positive orbit of a point out of C^{ss} remains out of C^{ss} until it leaves V.

For n large enough the points x_n belong to V.

As $\mathbb{R}X(x_n)$ tend to L, this implies that the point x_n, for n large, is contained in the cone C^{ss}.

In particular, the point x_n cannot belong to $W^u(\sigma)$. Therefore they admits negative iterates $y_n = \phi^{-t_n}(x_n)$ with the following property.

— $\phi^{-t}(x_n) \in V$ for all $t \in [0, t_n]$,

— $\phi^{-t_n-1}(x_n) \notin V$,

— $t_n \to +\infty$.

Up to consider a subsequence one may assume that y_n converges to a point y, and one can easily check that the point y belongs to $W^s(\sigma) \setminus \sigma$. Furthermore all the points y_n belong to $\Lambda_{X,U}$ so that $y \in \Lambda_{X,U}$.

We conclude the proof by showing that y belongs to $W^{ss}_{\sigma,U}$, which is a contradiction with the definition of the escaping strong stable manifold $W^{ss}_{\sigma,U}$. If $y \notin W^{ss}_{\sigma,U}$ then its positive orbit arrives to σ tangentially to the weaker stable spaces: in particular, there is $t > 0$ so that $\phi^t(y)$ does not belong to the cone C^{ss}.

Consider n large, in particular t_n is larger than t and $\phi^t(y_n)$ is so close to y that $\phi^t(y_n)$ does not belong to C^{ss}: this contradicts the fact that $x_n = \phi^{t_n}(y_n)$ belongs to C^{ss}.

We have proved $\Lambda_{\mathbb{P}}(X,U) \cap \mathbb{P}^{ss}_{\sigma,U} = \emptyset$; the proof that $\Lambda_{\mathbb{P}}(X,U) \cap \mathbb{P}^{uu}_{\sigma,U}$ is empty is analogous.

CQFD

As a consequence we get the following characterization of the central space of σ in U:

Lemma 20. *The central space $E^c_{\sigma,U}$ is the smallest center space containing $\Lambda_{\mathbb{P}}(X,U) \cap \mathbb{P}_\sigma$.*

Proof. The proof that $E^c_{\sigma,U}$ contains $\Lambda_{\mathbb{P}}(X,U) \cap \mathbb{P}_\sigma$ is very similar to the end of the proof of Lemma 19 and we just sketch it: by definition of the strong escaping manifolds, they admit a neighborhood of a fundamental domain which is disjoint from the maximal invariant set. This implies that any point in $\Lambda_{X,U}$ close to σ is contained out of arbitrarily large cones around the escaping strong direction. Therefore the vector X at these points is almost tangent to $E^c_{\sigma,U}$.

Assume now for instance that:

— $E^c_{\sigma,U} = E^s_i \oplus E^s_{i-1} \oplus \cdots \oplus E^s_1 \oplus E^u_1 \oplus \cdots \oplus E^u_j$: in particular $W^{ss}_{i+1}(\sigma)$ is the escaping strong stable manifold, and

— $\Lambda_{\mathbb{P}}(X,U) \cap \mathbb{P}_\sigma$ is contained in the smaller center space

$$E^s_{i-1} \oplus \cdots \oplus E^s_1 \oplus E^u_1 \oplus \cdots \oplus E^u_j.$$

We will show that the strong stable manifold $W^{ss}_i(\sigma)$ is escaping, contradicting the maximality of the escaping strong stable manifold $W^{ss}_{i+1}(\sigma)$. Otherwise, there is $x \in W^{ss}_i(\sigma) \setminus \{\sigma\} \cap \Lambda_{X,U}$. The positive orbit of x tends to σ tangentially to $E^s_k \oplus \cdots \oplus E^s_i$ and thus $X(\phi^t(x))$ for t large is almost tangent to $E^s_k \oplus \cdots \oplus E^s_i$: this implies that $\Lambda_{\mathbb{P}}(X,U) \cap \mathbb{P}_\sigma$ contains at least a direction in $E^s_k \oplus \cdots \oplus E^s_i$ contradicting the hypothesis. CQFD

Lemma 21. *Let U be a compact region. Let σ be a hyperbolic singular point in U, that has a continuation σ_Y for vector fields Y in a C^1-neighborhood of X. Then both escaping strong stable and unstable spaces $E^{ss}_{\sigma_Y,U}$ and $E^{uu}_{\sigma_Y,U}$ depend lower semi-continuously on Y.*

As a consequence the central space $E^c_{\sigma_Y,U}$ of σ_Y in U for Y depends upper semi-continuously on Y, and the same happens for its projective space $\mathbb{P}^s_{\sigma_Y,U}$.

Proof. We will make only the proof for the escaping strong stable space, as the proof for the escaping strong unstable space is identical.

As σ is contained in the interior of U, there is $\delta > 0$ and a C^1-neighborhood $\mathcal{U}$ of X so that, for any $Y \in \mathcal{U}$, one has:

— σ has a hyperbolic continuation σ_Y for Y;

— the finest dominated splitting of σ_X for X has a continuation for σ_Y which is a dominated splitting (but maybe not the finest);

— the local stable manifold of size δ of σ_Y is contained in U and depends continuously on Y

— for any strong stable space $E^{ss}(\sigma)$ the corresponding local strong stable manifold $W^{ss}(\sigma_Y)$ varies continuously with $Y \in \mathcal{U}$.

Let E^{ss} denote the escaping strong stable space of σ and $W^{ss}_\delta(\sigma)$ be the corresponding local strong stable manifold. We fix a sphere S_X embedded in the interior of $W^{ss}_\delta(\sigma)$, transverse to X and cutting every orbit in $W^{ss}_\delta(\sigma) \setminus \sigma$. By definition of escaping strong stable manifold, for every $x \in S_X$ there is $t(x) > 0$ so that $\phi^{t(x)}(x)$ is not contained in U.

As S_X is compact and the complement of U is open, there is a finite family $t_i, i = 0, \ldots, k$, an open covering $V_0, \ldots, V_k$ and a C^1-neighborhood $\mathcal{U}_1$ of X so that, for every $x \in U_i$ and every $Y \in \mathcal{U}_1$ the point $\phi^{t_i}_Y(x)$ does not belong to U.

For Y in a smaller neighborhood $\mathcal{U}_2$ of X, the union of the V_i cover a sphere $S_Y \subset W^{ss}_\delta(\sigma_Y, Y)$ cutting every orbit in $W^{ss}_\delta(\sigma_Y, Y) \setminus \sigma_Y$.

This shows that $W^{ss}_\delta(\sigma_Y, Y)$ is contained in the escaping strong stable manifold of σ_Y, proving the lower semi continuity.

CQFD

3.3 The extended maximal invariant set

We are now able to define the subset of $\mathbb{P}M$ which extends the lifted maximal invariant set and depends upper-semicontinuously on X.

3.3.1 The pre extended maximal invariant set

When the hypothesis of our problem gives us information about an open set of vector fields, for instance when we are talking about a robustly transitive sets, we define the following notion of extended maximal invariant set that we call *pre extended* and was introduced in [GLW]. Here the authors define a set of directions, that varies upper semi continuously with the vector field, and therefore all the robustness properties of the domination hold. The set is defined as follows:

Definition 19. Let U be a compact region and X a C^1 vector field. Let $\mathcal{U}$ be a neighborhood of X Then the set

$$\widetilde{\Lambda} = \overline{\{ < Y(x) > \in \mathbb{P}M \text{ such that } x \in U \cap Per(Y) \text{ and } Y \in \mathcal{U} \}}.$$

is called the *pre extended maximal invariant set of X in U*

From the definition, clearly the set $\Lambda_{\mathbb{P}}(X, U)$ is contained in $\widetilde{\Lambda}$, but we would like to extend $\Lambda_{\mathbb{P}}(X, U)$ to a set that can be defined without information on the perturbations of our vector field

3.3.2 The extended maximal invariant set

In a lot of situations however, we have no information about the surrounding vector fields, for instance when we are constructing examples. In this case it is more convenient to have an extended maximal invariant set that does not depend on the neighboring vector fields.

Definition 20. Let U be a compact region and X a vector field whose singular points are hyperbolic. Then the set

$$B(X, U) = \Lambda_{\mathbb{P}}(X, U) \cup \bigcup_{\sigma \in Sing(X) \cap U} \mathbb{P}^c_{\sigma,U} \subset \mathbb{P}M$$

is called the *extended maximal invariant set of X in U*

Proposition 22. *Let U be a compact region and X a vector field whose singular points are hyperbolic. Then the extended maximal invariant set $B(X, U)$ of X in U is a compact subset of $\mathbb{P}M$, invariant under the flow $\phi^t_{\mathbb{P}}$. Furthermore, the map $X \mapsto B(X, U)$ depends upper semi-continuously on X.*

Proof. First notice that the singular points of Y in U consists in finitely many hyperbolic singularities varying continuously with Y in a neighborhood of X. The extended maximal invariant set is compact as being the union of finitely many compact sets.

Let Y_n be a sequence of vector fields tending to X in the C^1-topology, and let $(x_n, L_n) \in B(Y_n, U)$. Up to considering a subsequence we may assume that (x_n, L_n) tends to a point $(x, L) \in \mathbb{P}M$ and we need to prove that (x, L) belongs to $B(X, U)$.

First assume that $x \notin Sing(X)$. Then, for n large, x_n is not a singular point for Y_n so that $L_n =< Y_n(x_n) >$ and therefore $L =< X(x) >$ belongs to $B(X, U)$, concluding.

Thus we may assume $x = \sigma \in Sing(X)$. First notice that, if for infinitely many n, x_n is a singularity of Y_n then L_n belongs to $\mathbb{P}^c_{\sigma_{Y_n},U}$. As $\mathbb{P}^c_{\sigma_Y,U}$ varies upper semi-continuously with Y, we conclude that L belongs to $\mathbb{P}^c_{\sigma_X,U}$, concluding.

So we may assume that $x_n \notin Sing(Y_n)$.

We fix a neighborhood V of σ endowed with coordinates, so that X (and therefore Y_n for large n) is very close to its linear part in V. Let $S_X \subset W^s_{loc}(\sigma)$ be a sphere transverse to X and cutting every orbit in $W^s_{loc}(\sigma) \setminus \{\sigma\}$, and let W be a small neighborhood of S_X.

First assume that, for infinitely many n, the point x_n does not belong $W^u(\sigma_{Y_n})$. There is a sequence $t_n > 0$ with the following property:

— $\phi^{-t}_{Y_n}(x_n) \in V$ for all $t \in [0, t_n]$

— $\phi^{-t_n}_{Y_n}(x_n) \in W$

— t_n tends to $+\infty$ as $n \to \infty$.

Up to considering a subsequence, one may assume that the points $y_n = \phi^{-t_n}_{Y_n}(x_n)$ tend to a point $y \in W^s(\sigma)$.

Claim. *The point y does not belong to $W^{ss}_{\sigma,U}$.*

Proof. By definition of the escaping strong stable manyfold, for every $y \in W^{ss}_{\sigma,U}$ there is t so that $\phi^t(y) \notin U$; thus $\phi^t_{Y_n}(y_n)$ do not belong to U for y_n close enough to y; in particular $y_n \notin \Lambda_{Y_n,U}$. CQFD

Thus y do not belong to $W^{ss}_{\sigma,U}$. Choosing $T > 0$ large enough, one gets that the line $< X(z) >$, $z = \phi^T(y)$, is almost tangent to $E^{cu} = E^c_{\sigma,U} \oplus E^{uu}_{\sigma,U}$. As a consequence, for n large, one gets that $< Y_n(z_n) >$, where $z_n = \phi^T_{Y_n}(y_n)$, is almost tangent to the continuation E^{cu}_n of E^c for σ_n, Y_n. As $x_n = \phi^{t_n - T}_{Y_n}(y_n)$, and as $t_n - T \to +\infty$, the dominated splitting implies that $L_n =< Y_n(x_n) >$ is almost tangent to E^{cu}_n.

This shows that L belongs to E^{cu}. Notice that this also holds if x_n belong to the unstable manifold of σ_{Y_n}.

Arguing analogously we get that L belongs to $E^{cs} = E^c_{\sigma,U} \oplus E^{ss}_{\sigma,U}$. Thus L belongs to $E^c_{\sigma,U}$ concluding. CQFD

The set the set $\widetilde{\Lambda}$ is the smallest of all possible sets containing the directions spanned by the vector field over the maximal invariant set. Therefore $\widetilde{\Lambda} \subset B(X, U)$. However, it is unknown to the author whether there is an example in which

$$\widetilde{\Lambda} \subsetneq B(X, U)$$

.

Chapter 4

Domination multisingular hyperbolicity and singular volume partial hyperbolicity

In this chapter we introduce the different hyperbolic structures that we are going to be dealing with. We define the reparametrizing cocycle, and prove that it is well defined. We prove Theorem 3 which relates de different ways of extending the linear Poincaré flow to the singularities. We define the notions of multisingular hyperbolicity and singular volume partial hyperbolicity, and prove some of their properties.

Let $\mathcal{A} = \{A^t(x)\}$ and $\mathcal{B} = \{B^t(x)\}$ be two linear cocycles on the same linear bundle $\pi\colon \mathcal{E} \to \Lambda$ and over the same flow ϕ^t on a compact invariant set Λ of a manifold M. We say that $\mathcal{B}$ is a *reparametrization* of $\mathcal{A}$ if there is a continuous map $h = \{h^t\}\colon \Lambda \times \mathbb{R} \to (0, +\infty)$ so that for every $x \in \Lambda$ and $t \in \mathbb{R}$ one has

$$B^t(x) = h^t(x)A^t(x).$$

The reparametrizing map h^t satisfies the cocycle relation

$$h^{r+s}(x) = h^r(x)h^s(\phi^r(x)),$$

and is called a *cocycle*.

One easily check the following lemma:

Lemma 23. *Let $\mathcal{A}$ be a linear cocycle and $\mathcal{B}$ be a reparametrization of $\mathcal{A}$. Then any dominated splitting for $\mathcal{A}$ is a dominated splitting for $\mathcal{B}$.*

Remark 21. — If h^t is a cocycle, then for any $\alpha \in \mathbb{R}$ the power $(h^t)^\alpha : x \mapsto (h^t(x))^\alpha$ is a cocycle.
— If f^t and g^t are cocycles then $h^t = f^t \cdot g^t$ is a cocycle.

A cocycle g^t is called a *coboundary* if there is a continuous function $g \colon \Lambda \to (0, +\infty)$ so that

$$g^t(x) = \frac{g(\phi^t(x))}{g(x)}.$$

A coboundary cocycle in uniformly bounded. Two cocycles f^t, h^t are called *cohomological* if $\frac{f^t}{h^t}$ is a coboundary.

Remark 22. The cohomological relation is an equivalence relation among the cocycle and is compatible with the product: if g_1^t and g_2^t are cohomological and h_1^t and h_2^t are cohomological then $g_1^t h_1^t$ and $g_2^t h_2^t$ are cohomological.

Lemma 24. *Let $\mathcal{A} = \mathcal{A}^t$ be a linear cocycle, and $h = h^t$ be a cocycle which is bounded. Then $\mathcal{A}$ is uniformly contracted (resp. expanded) if and only if the reparametrized cocycle $\mathcal{B} = h \cdot \mathcal{A}$ is uniformly contracted (resp. expanded).*

As a consequence one gets that the hyperbolicity of a reparametrized cocycle only depends on the cohomology class of the reparametrizing cocycle:

Corollary 25. *if g and h are cohomological then $g \cdot \mathcal{A}$ is hyperbolic if and only if $h \cdot \mathcal{A}$ is hyperbolic.*

4.1 Reparametrizing cocycle associated to a singular point

Let X be a C^1 vector field, ϕ^t its flows, and σ be a hyperbolic singularity of X. We denote by $\Lambda_X \subset \mathbb{P}M$ the union

$$\Lambda_X = \overline{\{\mathbb{R}X(x), x \notin Sing(X)\}} \cup \bigcup_{x \in Sing(X)} \mathbb{P}T_x M.$$

It can be shone easily that this set is upper semi-continuous, as in the case of $B(X, U)$ (see 22)

Lemma 26. *Λ_X is a compact subset of $\mathbb{P}M$ invariant under the flow $\phi^t_{\mathbb{P}}$, and the map $X \mapsto \Lambda_X$ is upper semi-continuous. Finally, if the singularities of X are hyperbolic then, for any compact regions one has $B(X, U) \subset \Lambda_X$.*

Let U_σ be a compact neighborhood of σ on which σ is the maximal invariant.

Let V_σ be a compact neighborhood of $Sing(X) \setminus \{\sigma\}$ so that $V_\sigma \cap U_\sigma = \emptyset$. We fix a ($C^1$) Riemmann metric $\|.\|$ on M so that

$$\|X(x)\| = 1 \text{ for all } x \in M \setminus (U_\sigma \cup V_\sigma).$$

Consider the map $h\colon \Lambda_X \times \mathbb{R} \to (0, +\infty)$, $h(L,t) = h^t(L)$, defined as follows:

— if $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma$ and $\phi^t(x) \notin U_\sigma$, then $h^t(L) = 1$ (if x and $\phi^t(x)$ are in $M \setminus (U_\sigma \cup V_\sigma)$, this would be a consecuence of our choice of metric, we ask that $h^t(L) = 1$ also if x and-or $\phi^t(x)$ are in V_σ);

— if $L \in \mathbb{P}T_xM$ with $x \in U_\sigma$ and $\phi^t(x) \notin U_\sigma$ then $L = \mathbb{R}X(x)$ and $h^t(L) = \frac{1}{\|X(x)\|}$;

— if $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma$ and $\phi^t(x) \in U_\sigma$ then $L = \mathbb{R}X(x)$ and $h^t(L) = \|X(\phi^t(x))\|$;

— if $L \in \mathbb{P}T_xM$ with $x \in U_\sigma$ and $\phi^t(x) \in U_\sigma$ but $x \neq \sigma$ then $L = \mathbb{R}X(x)$ and $h^t(L) = \frac{\|X(\phi^t(x))\|}{\|X(x)\|}$;

— if $L \in \mathbb{P}T_\sigma M$ then $h^t(L) = \frac{\|\phi^t_{\mathbb{P}}(u)\|}{\|u\|}$ where u is a vector in L.

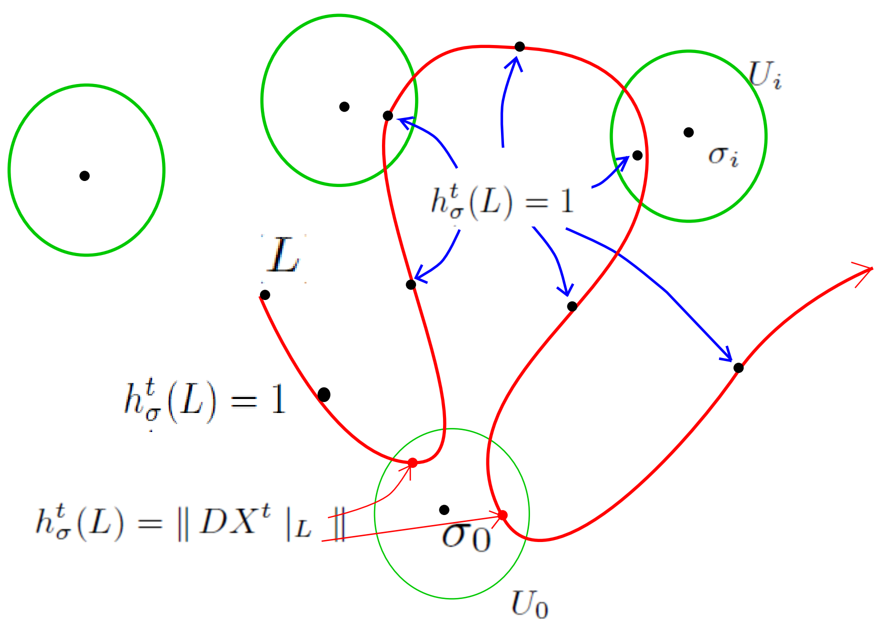


Figure 4.1 – the local cocycle h^t_σ associated to the singularity $\sigma = \sigma_0$

Lemma 27. *With the notation above, the map h is a (continuous) cocycle on Λ_X.*

Proof. The continuity of h comes from the continuity of the norm and the fact that the neighborhoods U_σ and V_σ do not intersect. Now we aim to show that h verifies the cocycle relation :

$$h^t(\phi^s_{\mathbb{P}}(L))h^s(L) = h^{t+s}(L)$$

— if $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma$, $\phi^s(x) \notin U_\sigma$ $\phi^{s+t}(x) \notin U_\sigma$, then $h^{t+s}(L) = h^t(\phi^s_{\mathbb{P}}(L))h^s(L) = 1$;

— Let $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma$, $\phi^s(x) \notin U_\sigma$ $\phi^{s+t}(x) \in U_\sigma$. Since $\|X(\phi^s(u))\| = 1$ then $h^s(L) = 1$ and,

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L)) &= \|X(\phi^t(\phi^s(u)))\| \\ &= \|X(\phi^{t+s}(u))\| \\ &= h^{t+s}(L), \end{aligned}$$

— if $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma$, $\phi^s(x) \in U_\sigma$ and $\phi^{t+s}(x) \notin U_\sigma$ then $L = \mathbb{R}X(x)$, $h^s(L) = \|X(\phi^s(x))\|$ and

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L) &= \frac{1}{\|X(\phi^s(x))\|}\|X(\phi^s(x))\| \\ &= 1 \\ &= h^{t+s}(L) \end{aligned}$$

— if $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma$, $\phi^s(x) \in U_\sigma$ and $\phi^{t+s}(x) \in U_\sigma$ then $L = \mathbb{R}X(x)$, $h^s(L) = \|X(\phi^s(x))\|$ and

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L) &= \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(\phi^s(x)\|}\|X(\phi^s(x))\|; \\ &= \|X(\phi^t(\phi^s(x))\| \\ &= \|X(\phi^{t+s}(x))\| \\ &= h^{t+s}(L) \end{aligned}$$

— if $L \in \mathbb{P}T_xM$ with $x \in U_\sigma$, $\phi^s(x) \notin U_\sigma$ $\phi^{s+t}(x) \notin U_\sigma$, then $h^t(\phi^s_{\mathbb{P}}(L)) = 1$

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L) &= \frac{1}{\|X(x)\|} \\ &= h^{t+s}(L), \end{aligned}$$

— Let $L \in \mathbb{P}T_xM$ with $x \in U_\sigma$, $\phi^s(x) \notin U_\sigma$ $\phi^{s+t}(x) \in U_\sigma$. Since $h^s(L) = \frac{1}{\|X(x)\|}$ then,

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L) &= \|X(\phi^t(\phi^s(x)))\| \frac{1}{\|X(x)\|} \\ &= \frac{\|X(\phi^{t+s}(x))\|}{\|X(x)\|} \\ &= h^{t+s}(L), \end{aligned}$$

— if $L \in \mathbb{P}T_xM$ with $x \in U_\sigma$, $\phi^s(x) \in U_\sigma$ and $\phi^{t+s}(x) \notin U_\sigma$ then $L = \mathbb{R}X(x)$, $h^t(\phi^s_{\mathbb{P}}(L)) = \frac{1}{\|X(\phi^s(x))\|}$ and

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L) &= \frac{\|X(\phi^s(x))\|}{\|X(x)\|} \frac{1}{\|X(\phi^s(x))\|} \\ &= \frac{1}{\|X(x)\|} \\ &= h^{t+s}(L). \end{aligned}$$

— if $L \in \mathbb{P}T_xM$ with $x \in U_\sigma$, $\phi^s(x) \in U_\sigma$ and $\phi^{t+s}(x) \in U_\sigma$ then $L = \mathbb{R}X(x)$,

$$\begin{aligned} h^t(\phi^s_{\mathbb{P}}(L))h^s(L) &= \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(\phi^s(x))\|} \frac{\|X(\phi^s(x))\|}{\|X(x)\|}; \\ &= \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(x)\|} \\ &= \frac{\|X(\phi^{t+s}(x))\|}{\|X(x)\|} \\ &= h^{t+s}(L) \end{aligned}$$

— if $L \in \mathbb{P}T_\sigma M$, let u be a vector in L, then

$$\begin{aligned} h^{t+s}(L) &= \frac{\|D\phi^{t+s}_{\mathbb{P}}(u)\|}{\|u\|}; \\ &= \frac{\|D\phi^{t+s}_{\mathbb{P}}(u)\|}{\|D\phi^s_{\mathbb{P}}(u)\|} \frac{\|D\phi^s_{\mathbb{P}}(u)\|}{\|u\|} \\ &= \frac{\|D\phi^t_{\mathbb{P}}(D\phi^s_{\mathbb{P}}(u))\|}{\|D\phi^s_{\mathbb{P}}(u)\|} \frac{\|D\phi^s_{\mathbb{P}}(u)\|}{\|u\|} \\ &= h^t(\phi^s_{\mathbb{P}}(L))h^s(L) \end{aligned}$$

CQFD

Lemma 28. *The cohomology class of a cocycle h defined as above, is independent from the choice of* *the metric $\|.\|$ and of the neighborhoods U_σ and V_σ.*

Proof. Let $\|.\|$ and $\|.\|'$ be two different metrics and 2 different sets of neighborhoods of σ and $Sing(X) \setminus \{\sigma\}$ such that:

— $V_\sigma \cap U_\sigma = \emptyset$.

— $V'_\sigma \cap U'_\sigma = \emptyset$.

— $\|X(x)\| = 1$ for all $x \in M \setminus (U_\sigma \cup V_\sigma)$,

— $\|X(x)\|' = 1$ for all $x \in M \setminus (U'_\sigma \cup V'_\sigma)$.

Additionally we ask that $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$. We will show at the end of the proof that this assumption does not make us lose generality.

We define h as above for the metric $\|.\|$ and h' as above for the metric $\|.\|'$. We define a function $g\colon B(X,U) \to (0,+\infty)$

— If $L \in \mathbb{P}T_xM$ with $x \notin V'_\sigma \cup V_\sigma$ then $g(L) = \frac{\|u\|'}{\|u\|}$ with u a non vanishing vector in L,

— and if $L \in \mathbb{P}T_xM$ with $x \in V'_\sigma \cup V_\sigma$, then $g(L) = 1$.

Claim. *The function $g\colon B(X,U) \to (0,+\infty)$ defined above is continuous*

Proof. Since $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$, the contituity in the boundary of $V \cup V'$ comes from the fact that $\|.\|$ and $\|.\|'$ are 1 out of $U \cup U'$. Also since $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$, the continuity of the norms $\|.\|$ and $\|.\|'$, and the fact that they are 1 out of $U \cup U'$, gives us the continuity in the boundary of $U \cup U'$. CQFD

The following claim will show us that the functions h and h' differ in a coboundary defined as $g^t(L) = \frac{g(D\phi^t_{\mathbb{P}}(u))}{g(u)}$.

Claim. *The functions h and h' are such that*
$h'^t(u) = h^t(u)\frac{g(D\phi^t_{\mathbb{P}}(u))}{g(u)}$.

Proof. — For the metric $\|.\|'$ and $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^t(x) \notin U_\sigma \cup U'_\sigma$, $g^t(L) = 1$. On the other side $h'^t(L) = 1$ as desired.

— If $L \in \mathbb{P}T_xM$ with $x \in U_\sigma \cup U'_\sigma$ and $\phi^t(x) \notin U_\sigma \cup U'_\sigma$ then $g^t(L) = \frac{\|u\|}{\|u\|'}$. Take $u = X(x)$ $h^t(L) = \frac{1}{\|X(x)\|}$;

$$h'^t(L) = h^t(L)\frac{\|X(x)\|}{\|X(x)\|'} .$$

— If $L \in \mathbb{P}T_xM$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^t(x) \in U_\sigma \cup U'_\sigma$ then $L = \mathbb{R}X(x)$ and take $u = X(x)$. $g^t(L) = \frac{\|D\phi^t_{\mathbb{P}}(u)\|'}{\|D\phi^t_{\mathbb{P}}(u)\|}$.Then since $h^t(L) = \|D\phi^t_{\mathbb{P}}(u)\|$,then

$$h'^t(L) = h^t(L)\frac{\|D\phi^t_{\mathbb{P}}(u)\|'}{\|D\phi^t_{\mathbb{P}}(u)\|} .$$

.

— If $L \in \mathbb{P}T_xM \cap B(X,U)$ with $x \in U_\sigma \cup U'_\sigma$ and $\phi^t(x) \in U_\sigma \cup U'_\sigma$. Take $u = X(x)$, then

$$g^t(L)\frac{\|D\phi^t_{\mathbb{P}}(u)\|'\|u\|}{\|D\phi^t_{\mathbb{P}}(u)\|\|u\|'},$$

and $h^t(L) = \frac{\|D\phi^t_{\mathbb{P}}(u)\|}{\|u\|}$. So $h'^t(L) = h^t(L)g^t(L)$

CQFD

Now in order to finish the proof we need to show that assuming the condition that the norms where such that $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$ does not make us lose generality. For this, suppose we started with any other norm $\|.\|''$ and that there exist two neighborhood such that.

— $V''_\sigma \cap U''_\sigma = \emptyset$.

— $\|X(x)\|'' = 1$ for all $x \in M \setminus (U''_\sigma \cup V''_\sigma)$.

Let us choose a smaller neighborhood $V'_\sigma \subset V''_\sigma$. This satisfies $V'_\sigma \cap U''_\sigma = \emptyset$. Analogously $U'_\sigma \subset U''_\sigma$ will satisfy $V''_\sigma \cap U'_\sigma = \emptyset$. Now if we choose this neighborhoods V' and U' as small as we want, and a norm norm $\|.\|'$ such that $\|X(x)\|' = 1$ for all $x \in M \setminus (U''_\sigma \cup V''_\sigma)$. the claims above implies that the corresponding h'' and h' differ in a coboundary. The open sets V' and U' can be chosen so that $V'_\sigma \cap U_\sigma = \emptyset$ and $V'_\sigma \cap U_\sigma = \emptyset$. Therefore h and h' differ in a coboundary and that implies that h'' and h differ in a coboundary. CQFD

We denote by $[h(X,\sigma)]$ the cohomology class of any cocycle defined as above.

Lemma 29. *Consider a vector field X and a hyperbolic singularity σ of X. Then there is a C^1-neighborhood $\mathcal{U}$ of X so that σ has a well defined hyperbolic continuation σ_Y for Y in $\mathcal{U}$ and for any $Y \in \mathcal{U}$ there is a map $h_Y \colon \Lambda_Y \times \mathbb{R} \to (0,+\infty)$ so that*

— *for any Y, h_Y is a cocycle belonging to the cohomology class $[h(Y,\sigma_Y)]$*

— *h_Y depends continuously on Y: if $Y_n \in \mathcal{U}$ converge to $Z \in \mathcal{U}$ for the C^1-topology and if $L_n \in \Lambda_{Y_n}$ converge to $L \in \Lambda_Z$ then there are representatives of the homological classes of $[h(Y_n,\sigma_{Y_n})]$ such that $h^t_{Y_n}(L_n)$ tends to $h^t_Z(L)$ for every $t \in \mathbb{R}$; furthermore this convergence is uniform in $t \in [-1,1]$.*

Proof. The manifold M is endowed with a Riemmann metric $\|.\|$. We fix the neighborhoods U_σ and V_σ for X and $\mathcal{U}$ is a C^1-neighborhood of X so that σ_Y is the maximal invariant set for Y in U_σ and $Sing(Y) \setminus \{\sigma_Y\}$ is contained in the interior of V_σ. Up to shrink $\mathcal{U}$ if necessary, we also assume that there are compact neighborhoods $\tilde{U}_\sigma$ of σ_Y contained in the interior of U_σ and $\tilde{V}_\sigma$ of $Sing(Y) \setminus \{\sigma_Y\}$ contained in the interior of V_σ.

We fix a continuous function $\xi\colon M \to [0,1]$ so that $\xi(x) = 1$ for $x \in M \setminus (U_\sigma \cup V_\sigma)$ and $\xi(x) = 0$ for $x \in \tilde{U}_\sigma \cup \tilde{V}_\sigma$.

For any $Y \in \mathcal{U}$ we consider the map $\eta_Y\colon M \to (0, +\infty)$ defined by

$$\eta_Y(x) = \frac{\xi(x)}{\|X(x)\|} + 1 - \xi(x).$$

This map is a priori not defined on $Sing(Y)$ but extends by continuity on $y \in Sing(Y)$ by $\eta_Y(y) = 1$, and is continous.

This maps depends continuously on Y. Now we consider the metric $\|.\|_Y = \eta_Y \|.\|$. Note that $\|Y(x)\|_Y = 1$ for $x \in M \setminus (U_\sigma \cup V_\sigma)$.

Now h_Y is the cocycle built at lemma 28 for U_σ, V_σ and $\|.\|_Y$. CQFD

Notice that, according to Remark 22, if $\sigma_1, \dots, \sigma_k$ are hyperbolic singularities of X the homology class of the product cocycle $h^t_{\sigma_1} \cdots h^t_{\sigma_k}$ is well defined, and admits representatives varying continuously with the flow.

4.2 Extension of the Dominated splitting

4.2.1 Relating the center space of the singularities with the dominated splitting on $\widetilde{\Lambda}$

Let us consider a singularity $\sigma \in C$. We consider the following splitting of its tangent space:

$$E^{ss} \oplus E^c \oplus E^{uu} \,,$$

noting the stable escaping, the unstable escaping and the center spaces. We can suppose the singularities to be hyperbolic and that their finest hyperbolic splitting is into one or two dimensional spaces, this are generic conditions. Once we obtain a dominated splitting in this setting, we can conclude later for the non hyperbolic case by using corollary 12. Let us consider the Lyapunov exponents of the hyperbolic splitting restricted to the center space:

$$\lambda_1 < \cdots < \lambda_l$$

and the Lyapunov spaces associated to them

$$E^c = E_1 \oplus \cdots \oplus E_l \,.$$

Note that it follows from the definition of center space that $\widetilde{\Lambda} \subset B(X, U)$ and from Theorem 18 we have that $\Lambda_{\mathbb{P}}(X, U) \subset \widetilde{\Lambda}$.

Lemma 30. *Let us consider a singularity σ where the tangent space splits into*

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu} ,$$

the escaping spaces and the center space. We consider as well the finest lyapunov splitting over the singularity of the center space is $E^c = E_1 \oplus \cdots \oplus E_l$. If the dimension of the center space is locally constant then

$$\pi_{\mathbb{P}}(E_i) \cap \widetilde{\Lambda} \neq \emptyset$$

for all the E_i, Lyapunov spaces of the hyperbolic splitting. More over

$$\pi_{\mathbb{P}}(E_i) \subset \widetilde{\Lambda}$$

for all the E_i, Lyapunov spaces of the hyperbolic splitting.

Proof. Let us consider $\pi_{\mathbb{P}}(E_1)$ in $\mathbb{P}^c_\sigma$. By definition of center space, there is an orbit γ_1 tangent to $E^{ss} \oplus E_1$, that is not tangent to E^{ss}. This implies that $\pi_{\mathbb{P}}(E_1) \cap \widetilde{\Lambda} \neq \emptyset$

We consider a linear neighborhood of σ. First we perturb X to a vector field Y' that is Kupka-Smale. The vector field Y' is in the hypothesis of the lemma as well since this assumptions are robst By 17 we perturb Y' to Y so that γ_1 is a homoclinic connection of the singularity and without changing the fact that γ_1 becomes tangent to $E^{ss} \oplus E_1$ as it approaches the singularity. Now we perturb Y to Y_1 braking the homoclinic connection in the direction of E_2 so thats no longer tangent to $E^{ss} \oplus E_1$ but is tangent to $E^{ss} \oplus E_1 \oplus E_2$. The domination implies that the orbit will become tangent to E_2 as it approaches σ. We can do this perturbation so that γ_1 remains the same out of the linear neighborhood of the singularity and so that the α-limit also remains the same. Therefore, γ_1 still belongs to $\Lambda(Y_1, U)$. By Lema 18 there is a sequence of vector fields Y_n and periodic orbits γ_n having γ_1 in their limit. Therefore $\pi_{\mathbb{P}}(E_2) \cap \widetilde{\Lambda} \neq \emptyset$. We can continue this process for all $1 \leq i \leq l$.

If the center space splits into only one or two dimensional spaces let us take $L \in \pi_{\mathbb{P}}(E_i) \cap \widetilde{\Lambda}$ where E_i is two dimensional with complex lyapunov exponents . Since $\widetilde{\Lambda}$ then the orbit of L under $\phi^t_{\mathbb{P}}$, that we note $O(L)$, is such that $O(L) \subset \widetilde{\Lambda}$. Since E_i has complex lyapunov exponents, the direction L is not invariant and $O(L)$ covers all directions of E_i and therefore $\pi_{\mathbb{P}}(E_i) \subset \Lambda_{\mathbb{P}}(X, U)$.

If the center space of X does not split into only one or two dimensional spaces but the dimension of the center space is loccaly constant in $\mathcal{U}$, a C^1 neighborhood of X, there is $Y \in \mathcal{U}$ such that center space splits into only one or two dimensional spaces and with the same dimension of the center space. Therefore $\pi_{\mathbb{P}}(E_i) \subset \widetilde{\Lambda}$. CQFD

Lemma 31. *We consider X a flow with the following properties:*

We consider a direction L_1 in $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$, such that $L_1 =< u >$ where u belongs to some E^1 and L_l in $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$, such that $L_l =< v >$ where v belongs to some E^l.

— *A singularity σ and the finest hyperbolic splitting over the singularity*

$$T_\sigma M = E^{ss} \oplus E_1 \oplus \cdots \oplus E_i \oplus \cdots \oplus E_l \oplus E^{uu}.$$

— *C_σ the chain class of σ with U a filtrating neighborhood and Λ is the maximal invariant set in U*

— *The dimension of E^c is locally constant.*

Then for any C^1 open set $\mathcal{U}$ of X, there is Y in $\mathcal{U}$ such that there is a homoclinic orbit γ, that approaches the singularity tangent to the L_1 direction and for the past, tangent to L_l.

Proof. First let us note that if an orbit contained in the stable or unstable manifold of the singularity, escapes a neighborhood U of C_σ then it also escapes $\overline{U}$. By Lema 21 we have that there is a C^1 open and dense set such that the dimension of the central space is locally constant.

Let us also observe that if there exist an open set U of C_σ such that an orbit contained in the stable or unstable manifold of the singularity, escapes U, then the orbit also escapes a basis of open sets around C_σ, therefore the orbits that do not escape for any U are in C_σ. Let us consider the fines hyperbolic decomposition of the center space for this vector field:

$$E^c = E_1 \oplus \cdots \oplus E_l.$$

By definition, there is an orbit in the stable manifold tangent to $E^{ss} \oplus E_1$ that is contained in C_σ and there is an orbit in the unstable manifold tangent to $E_l \oplus E^{uu}$ that is contained in C_σ. In the open set around X such that the dimension of the center space is constant, we choose Y such that it is Kupka-Smale , and the orbit in the stable manifold tangent to $E^{ss} \oplus E_1$ approaches the singularity in the direction of E_1 and the orbit in the unstable manifold tangent to $E^l \oplus E_{uu}$ approaches the singularity in the direction of E_l. By theorem 17 we can get another vector field Y_1 arbitrarily close to Y that has an homoclinic orbit Γ such that it approaches the singularity in the direction of E_1 for the future and in the direction of E_l for the past (observe that for Y_1 the dimension of the center space is the same as for X).

CQFD

Lemma 32. *We consider X a flow with a hyperbolic singularity σ where the tangent space splits into*

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu} ,$$

the escaping spaces and the center space. We consider as well the finest hyperbolic splitting over the singularity of

— $E^c = E_1 \oplus \cdots \oplus E_l$
— $E^{ss} = E_{s1} \oplus \cdots \oplus E_{sk}$
— $E^{uu} = E_{u1} \oplus \cdots \oplus E_{ur}$

Let L be a direction in $B(X,U) \cap \mathbb{P}_\sigma M$, then: $\mathcal{N}_L = E^{ss} \oplus \pi_L(E^c) \oplus E^{uu}$, is a dominated splitting over $B(X,U) \cap \mathbb{P}_\sigma M$.

In addition, let L be such that $L =< u >$ where u belongs to some E^i. Then

$$\mathcal{N}_L = E_{s1} \oplus \cdots \oplus E_{sk} \oplus E_1 \oplus \cdots \oplus \pi_L(E_i) \oplus \cdots \oplus E_l \oplus E_{u1} \oplus \cdots \oplus E_{ur} .$$

is the finest dominated splitting over the orbit of L.

Proof. We begin by considering a norm in $T_\sigma M$ so that all the spaces in the decomposition are normal. Let L be a direction in $\pi_{\mathbb{P}}(E^c)$, if L does not belong to E_1 or E_l then $\mathcal{N}_L = E^{ss} \oplus \pi_L(E^c) \oplus E^{uu}$, is a dominated splitting over the orbit of L..

Let L be a direction in $B(X,U) \cap \mathbb{P}_\sigma M$, and such that $L =< u >$ where u belongs to some E^i. If E_i is one dimensional, then

$$\mathcal{N}_L = E_{s1} \oplus \cdots \oplus E_1 \oplus \cdots \oplus E_{i-1} \oplus E_{i+1} \oplus \cdots \oplus E_l \oplus \cdots \oplus E_{ur} .$$

is the finest dominated splitting over the orbit of L.

Let us recall that E_i has Lyapunov exponents of the same modulus λ. For any $L \in \pi_{\mathbb{P}}(E_i)$ we choose v such that $< v >= L$. We take $w \in \pi_L(E_i)$ and we include $w \in E_i$ with the canonical inclusion.

The angle between $D\phi^t(w)$ and $D\phi^t(v)$ cannot go to 0 with t or else there would be some direction in E_i that dominates the others. Therefore the norm of $\pi_{\phi^t_{\mathbb{P}}(L)}$ is bounded away from 0 in restriction to $D\phi^t(\pi_L(E_i))$. The norm of a projection is always bounded from above by 1. Since $D\phi^t$ expands exponentially the norm of w by a factor λ, then so does ψ^t. Therefore

$$\mathcal{N}_L = E_{s1} \oplus \cdots \oplus E_1 \oplus \cdots \oplus \pi_L(E_i) \oplus \cdots \oplus E_l \oplus \cdots \oplus E_{ur} ,$$

is a dominated splitting.

CQFD

Remark 23. The same is true for $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$, since it is a compact invariant subset of $B(X, U) \cap \mathbb{P}_\sigma M$

In this section we suppose that the extended linear Poincaré flow over $\widetilde{\Lambda}$ has a dominated splitting,

$$\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F ,$$

where L is a direction in $\widetilde{\Lambda}$.

We call $\pi_L : T_x M \to \mathcal{N}_L$ where $L \in \mathbb{P}_x M$ the projection over the normal space at a given direction L.

The following lemma is very similar to lemma 4.3 in [GLW].

Lemma 33. *Let X be a vector field having singular chain class C_σ, we consider a filtrating neighborhood U and the maximal invariant set Λ in U. We denote $S = Sing(X) \cap U$ and we suppose that*

- *there is $\sigma \in S$ that is hyperbolic*
- *the dimension of the center space of σ is locally constant.*
- *there exist a C^1-neighborhood $\mathcal{U}$ such that the extended linear Poincaré flow over $\widetilde{\Lambda}$ has a dominated splitting,*

  $$\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F ,$$

  where L is any direction in $\widetilde{\Lambda}$.

Let L be a direction in $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$, Then,

$$\pi_L(E^c_\sigma) \subset \mathcal{N}^E_L ,$$

or

$$\pi_L(E^c_\sigma) \subset \mathcal{N}^F_L .$$

Proof. Since σ is hyperbolic we can suppose that the tangent space of σ splits into

$$T_\sigma = E^{ss} \oplus E^c \oplus E^{uu} ,$$

the escaping spaces and the center space. We consider as well the finest hyperbolic splitting over the singularity of $E^c = E_1 \oplus \cdots \oplus E_l$.

Let us suppose that $dim(\mathcal{N}^E_L) = n$, then if the $dim(E^{ss}) \geq n$, then $\mathcal{N}^E_L \subset \pi_L(E^{ss})$. This implies that $\pi_L(E^c_\sigma) \subset \mathcal{N}^F_L$.

Suppose now that $dim(E^{ss}) < dim(\mathcal{N}^E_{L_1})$ from Lema 31 for every L Then $\pi_L(E_1) \subset \mathcal{N}^E_L$.

We suppose by contradiction that there exist a direction L such that $\mathcal{N}_L^E$ contains some vector $v \in \pi_L(E_1 \oplus \cdots \oplus E_l)$ if $v \in E_l$ then Lema 32 $\mathcal{N}_L^E \cap E_l = \emptyset$. Then the dimension of $\mathcal{N}_L^E$ is lesser than $dim(\pi_L(E^s s \oplus E^c)$ for any L in $\widetilde{\Lambda}$.

If $L \notin E_l$ then let us suppose that $v \in E_l$ then $\mathcal{N}_L^E \cap E_l = \emptyset$.

For the C^1-neighborhood of the definition of $\widetilde{\Lambda}$, $\mathcal{U}$, we can find a vector field Y' that is Kupka-Smale. Since L also belongs to $\widetilde{\Lambda} \cap \mathbb{P}_\sigma M$, from Lema 31, we can perturb Y' and find a vector field $Y \subset \mathcal{U}$ having a homoclinic orbit γ such that it approaches the singularity σ tangent to L and it approaches the singularity for the past, tangent to a direction L_u in E_l We consider now a linearized neighborhood of the singularity that we call U_σ, and choose two regular points x, y such that $x \in W^s_{loc}\sigma \cap \gamma$ and $y \in W^u_{loc}\sigma \cap \gamma$ Then we can choose $xn \to x$ and $y_n \to y$, such that $\phi_{t_n}(x_n) = y_n$ and $\{ \phi_t(x_n) \text{ for every} 0 \leq t \leq t_n \}$ is tangent to $E_1 \oplus E_l$, note that we are in the linearized neighborhood so actually $\{ \phi_t(x_n) \text{ for every} 0 \leq t \leq t_n \} \subset E_1 \oplus E_l$. And for n big enough we can still suppose that the segment of orbit from x_n to y_n is in U_σ.

We can now find a p_n in the orbit of x_n, $O(x_n)$ satisfying $p_n \to \sigma$ and if L_n is such that $L_n =< X_n(x_n) >$, then the upper limit of $L_n \subset \widetilde{\Lambda}$, (i.e. all limit points of L_n are in $\widetilde{\Lambda}$). In fact the limit points of L_n are a one dimensional subspace of $E_1 \oplus E_l$.

By an appropriate choice of p_n and taking subsequence when necessary, we may assume that $L_n \to L \in E_1 \oplus E_l \setminus (E_1 \cup E_l)$. We can take a unit vector v such that

— $L =< v >$.

— $v = v_1 + v_l$ with $v_1 \in E_1$ and $v_l \in E_l$

We define w as $w = v_1 - v_l$, and $w \perp v$. This implies that $w \in \mathcal{N}_L$. Since E_1 is contracting and E_l is expanding, we have that $\phi^t_{\mathbb{P}}(L)$ goes to E_l for positive t and since $w \in \mathcal{N}_L \cap E_1 \oplus E_l$ $\psi^t_{\mathcal{N}}(w)$ goes to E_1.

On the other side we have that $\phi^t_{\mathbb{P}}(L)$ goes to E_1 for negative t and since $w \in \mathcal{N}_{L_n} \cap E_1 \oplus E_l$ $\psi^t_{\mathcal{N}}(w)$ goes to E_l. Now we consider the following two cases:

1. $w \in \mathcal{N}_L^E$
2. $w \in \mathcal{N}_L^F$

For the first case, for a negative t case, $\psi^t_{\mathcal{N}}(w)$ goes to E_l. We have that $w \in \mathcal{N}_L^E$ and $\phi^t_{\mathbb{P}}(L) \to E_1$ but as t goes to minus infinity, $\psi^t_{\mathcal{N}}(w)$ goes to E_l, and $E_l \cap \mathcal{N}_L^E$ is empty. This is a contradiction

For the second case, and a positive t $\psi^t_{\mathcal{N}}(w)$ goes to E_l, but We have that $w \in \mathcal{N}_L^F$ but as t goes to infinity, $\psi^t_{\mathcal{N}}(w)$ goes to E_1, and $E_1 \subset \mathcal{N}_L^E$. this is a contradiction

CQFD

Corollary 34. *Let X be a vector field having a maximal invariant set Λ in U suppose as well that*

— *all singularities in* $S = Sing(X) \cap U$ *are hyperbolic,*
— *The dimension of all center spaces over the singularities are locally constant.*
. Then the pre extended maximal invariant set $\widetilde{\Lambda}$ *has a dominated splitting if and only if the extended maximal invariant set* $B(X, U)$ *has a dominated splitting of the same dimension.*

Proof. Suppose that $B(X, U)$ has a dominated splitting, then $\widetilde{\Lambda}$ has a dominated splitting of the same dimension since it is a compact invariant subset. Suppose that there is a dominated splitting of the normal bundle in $\widetilde{\Lambda}$

$$\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F ,$$

Then according to the previous lemma we have 2 possibilities

$$\pi_L(E^c_\sigma) \subset \mathcal{N}^E_L ,$$

or

$$\pi_L(E^c_\sigma) \subset \mathcal{N}^F_L .$$

The tangent space of σ splits into

$$T_\sigma = E^{ss} \oplus E^c \oplus E^{uu} ,$$

the escaping spaces and the center space. We consider as well the finest hyperbolic splitting over the singularity of

— $E^c = E_1 \oplus \cdots \oplus E_l$
— $E^{ss} = E_{s1} \oplus \cdots \oplus E_{sk}$
— $E^{uu} = E_{u1} \oplus \cdots \oplus E_{ur}$

So if we are in the case where $\pi_L(E^c_\sigma) \subset \mathcal{N}^E_L$, lemma 32 implies that there exist an i such that

$$\mathcal{N}^F_L = E_{ui} \oplus \cdots \oplus E_{ur}$$

and

$$\mathcal{N}^E_L = E_{ss} \oplus E^c \oplus E_{u1} \cdots \oplus E_{ui-1}$$

. This same dominated splitting can be defined for any $L \in B(X, U)$. The other case is analogous.

We can do the same for every singularity in Λ CQFD

Corollary 35. *Let* X *be a vector field having a maximal invariant set* Λ *in* U*. Suppose as well that*
— *all singularities in* $S = Sing(X) \cap U$ *are hyperbolic,*

— *The dimension of all center spaces over the singularities are locally constant.*

. Then the pre extended maximal invariant set has a dominated splitting of the normal bundle $\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F$ such that the extended linear Poincaré flow contracts uniformly or in volume $\mathcal{N}^E$ if and only if hen the extended maximal invariant set has a dominated splitting of the normal bundle $\mathcal{N}_L = \mathcal{N}^E \oplus \mathcal{N}^F$ such that the extended linear Poincaré flow contracts uniformly or in volume $\mathcal{N}^E$,

Proof. From corollary 34 the dominated splitting in $\widetilde{\Lambda}$, extends to $B(X,U)$.

The tangent space of σ splits into

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu} \,,$$

the escaping spaces and the center space. We consider as well the finest Lyapunov splitting over the singularity of $E^c = E_1 \oplus \cdots \oplus E_l$, from lemma 30 we have that for every $i \in \{1 \ldots l\}$, $\pi_{\mathbb{P}}(E_i) \cap \widetilde{\Lambda} \neq \emptyset$. Then let us suppose that $\mathcal{N}^E$ contracts volume the extended linear Poincaré flow over $\widetilde{\Lambda}$. If $L_1 \in \pi_{\mathbb{P}}(E_i)$ is such that $\mathcal{N}^E_{L_1}$ contracts volume for the extended linear Poincaré flow, then $\mathcal{N}^E_L$ contracts volume for the extended linear Poincaré flow. This is because all Lyapunov exponents in E_i are of the same modulus. Since $\pi_{\mathbb{P}}(E_i) \cap \widetilde{\Lambda} \neq \emptyset$, then $\mathcal{N}^E_{L_1}$ contracts volume for the orbit of $L \in B(X,U) \cap \pi_{\mathbb{P}}(E_i)$.

Then we consider $L \in B(X,U)$ and u a vector in the direction of L. We write u in coordinates of the center space $u = (u_1, \ldots, u_i, \ldots, u_j \ldots u_l)$. We suppose as well that u_i is the first non zero coordinate of u and u_j is the last. Domination implies that for t sufficiently negatively large $\phi^t_{\mathbb{P}}(L)$ is in a small cone around $\pi_{\mathbb{P}}(E_i)$ and remains there, there after. For the future $\phi^t_{\mathbb{P}}(L)$ is in a small cone around $\pi_{\mathbb{P}}(E_j)$ and remains there, there after. Since the contraction and expansion rates extend to the cones around $\pi_{\mathbb{P}}(E_j)$ and $\pi_{\mathbb{P}}(E_i)$, and the orbit is outside of this cones only finite time, we get our conclusion. The same is true for uniform contraction and it is analogous for expansions. CQFD

With this last corollary we complete the proof of theorem 3

Corollary 36. *Let X be a vector field having a maximal invariant set Λ in U. Suppose as well that*

— *all singularities in $S = Sing(X) \cap U$ are hyperbolic,*

— *The dimension of all center spaces over the singularities are locally constant.*

.

— *There is a subset $S_F \subset S$ so that the reparametrized cocycle $h^t_F \psi^t_{\mathcal{N}}$ is uniformly contracted (expanded) in restriction to the a bundle of the domination over $\widetilde{\Lambda}$ where h_F denotes*

$$h_F = \Pi_{\sigma \in S_F} h_\sigma.$$

if and only if the same is true for that bundle in the domination over $B(X,U)$.

— *There is a subset $S_F \subset S$ so that the reparametrized cocycle $h_F^t \psi_{\mathcal{N}}^t$ is contracts (expands) volume in restriction to the a bundle of the domination over $\widetilde{\Lambda}$ where h_F denotes*

$$h_F = \Pi_{\sigma \in S_F} h_\sigma.$$

if and only if the same is true for that bundle in the domination over $B(X,U)$.

There is a subset $S_F \subset S$ so that the reparametrized cocycle $h_F^t \psi_{\mathcal{N}}^t$ is uniformly contracted (expanded) in restriction to the a bundle of the domination over $\widetilde{\Lambda}$ where h_F denotes

$$h_F = \Pi_{\sigma \in S_F} h_\sigma.$$

if and only if the same is true for that bundle in the domination over $B(X,U)$.

The proof is analogous to the previous one.

We are now ready for defining our notion of multisingular hyperbolicity.

4.3 Definition of multisingular hyperbolicity

Definition 24. Let X be a C^1-vector field on a compact manifold and let U be a compact region. One says that X is *multisingular hyperbolic* on U if

1. Every Sing of X in U is hyperbolic. We denote $S = Sing(X) \cap U$.
2. The restriction of the extended linear Poincaré flow $\{\psi_{\mathcal{N}}^t\}$ to the extended maximal invariant set $B(X,U)$ admits a dominated splitting $\mathcal{N}_L = E_L \oplus F_L$.
3. There is a subset $S_E \subset S$ so that the reparametrized cocycle $h_E^t \psi_{\mathcal{N}}^t$ is uniformly contracted in restriction to the bundles E over $B(X,U)$ where h_E denotes

$$h_E = \Pi_{\sigma \in S_E} h_\sigma.$$

4. There is a subset $S_F \subset S$ so that the reparametrized cocycle $h_F^t \psi_{\mathcal{N}}^t$ is uniformly expanded in restriction to the bundles F over $B(X,U)$ where h_F denotes

$$h_F = \Pi_{\sigma \in S_F} h_\sigma.$$

Remark 25. The subsets S_E and S_F are not necessarily uniquely defined, leading to several notions of multisingular hyperbolicity. We can also modify slightly this definition allowing to consider the product of power of the h_σ. In that case $\tilde{h}_E$ would be on the form

$$h_E^t = \Pi_{\sigma \in S_E} (h_\sigma^t)^{\alpha_E(\sigma)}$$

where $\alpha_E(\sigma) \in \mathbb{R}$.

Our first result in this section is now

Theorem 37. *Let X be a C^1-vector field on a compact manifold M and let $U \subset M$ be a compact region. Assume that X is multisingular hyperbolic on U. Then X is a star flow on U, that is, there is a C^1-neighborhood $\mathcal{U}$ of X so that every periodic orbit contained in U of a vector field $Y \in \mathcal{U}$ is hyperbolic. Furthermore $Y \in \mathcal{U}$ is multisingular hyperbolic in U.*

Proof. Recall that the extended maximal invariant set $B(Y, U)$ varies upper semicontinuously with Y for the C^1-topology . Therefore, according to Proposition 10 there is a C^1-neighborhood $\mathcal{U}_0$ of X so that, for every $Y \in \mathcal{U}_0$ the extended linear Poincré flow $\psi^t_{\mathcal{N},Y}$ admits a dominated splitting $E \oplus_< F$ over $B(Y, U)$, whose dimensions are independent of Y and whose bundles vary continuously with Y.

Now let S_E and S_F be the sets of singular point of X in the definition of singular hyperbolicity. Now Lemma 29 allows us to choose two families of cocycles $h^t_{E,Y}$ and $h^t_{F,Y}$ depending continuously on Y in a small neighborhood $\mathcal{U}_1$ of X and which belongs to the product of the cohomology class of cocycles associated to the singularities in S_E and S_F, respectively. Thus the linear cocycles

$$h^t_{E,Y} \cdot \psi^t_{\mathcal{N},Y}|_{E,Y}, \quad \text{over } B(Y, U)$$

varies continuously with Y in $\mathcal{U}_1$, and is uniformly contracted for X. Thus, it is uniformly contracted for Y in a C^1-neighborhood of X.

One shows in the same way that

$$h^t_{F,Y} \cdot \psi^t_{\mathcal{N},Y}|_{F,Y}, \quad \text{over } B(Y, U)$$

is uniformly expanded for Y in a small neighborhood of X.

We just prove that there is a neighborhood $\mathcal{U}$ of X so that $Y \in \mathcal{U}$ is multisingular hyperbolic in U.

Consider a (regular) periodic orbit γ of Y ant let π be its period. Just by construction of the cocycles h_E and h_F, one check that

$$h_E^\pi(\gamma(0)) = h_F^\pi(\gamma(0)) = 1.$$

One deduces that the linear Poincaré flow is uniformly hyperbolic along γ so that γ is hyperbolic, ending the proof.

CQFD

4.3.1 The multisingular hyperbolic structures over a singular point

The aim of this section is next proposition

Proposition 38. *Let X be a C^1-vector field on a compact manifold and $U \subset M$ a compact region. Assume that X is multisingular hyperbolic in U and let i denote the dimension of the stable bundle of the reparametrized extended linear Poincaré flow.*

Let σ be a singularity of X. Then

- *either at least one entire invariant (stable or unstable) manifold of σ is escaping from U.*
- *or σ is Lorenz like, more precisely*
  - *either the stable index is $i+1$, the center space $E^c_{\sigma,U}$ contains exactly one stable direction E^s_1 (dim $E^s_1 = 1$) and $E^s_1 \oplus E^u(\sigma)$ is sectionally dissipative; in this case $\sigma \in S_F$.*
  - *or the stable index is i, the center space $E^c_{\sigma,U}$ contains exactly one unstable direction E^u_1 (dim $E^u_1 = 1$) and $E^s(\sigma) \oplus E^u_1$ is sectionally contracting; then $\sigma \in S_E$.*

Note that in the first case of this proposition the class of the singularity must be trivial. If it was not, the regular orbits of the class that accumulate on σ, entering U, would accumulate on an orbit of the stable manifold. Therefore the stable manifold could not be completely escaping. The same reasoning holds for the unstable manifold.

Let $E^s_k \oplus_< \cdots \oplus_< E^s_1 \oplus_< E^u_1 \oplus_< \cdots \oplus_< E^u_\ell$ be the finest dominated splitting of the flow over σ. For the proof, we will assume, in the rest of the section that the class of σ is not trivial, and therefore we are not in the first case of our previous proposition. In other word, we assume that there are $a > 0, b > 0$ so that

$$E^c_{\sigma,U} = E^s_a \oplus_< \cdots \oplus_< E^s_1 \oplus_< E^u_1 \oplus_< \cdots \oplus_< E^u_b.$$

We assume that X is multisingular hyperbolic of s-index i and we denote by $E \oplus_< F$ the corresponding dominated splitting of the extended linear Poincaré flow over $B(X, U)$.

Lemma 39. *et X be a C^1-vector field on a compact manifold and $U \subset M$ a compact region. Assume that X is multisingular hyperbolic in U and let i denote the dimension of the stable bundle of the reparametrized extended linear Poincaré flow.*

let σ be a singularity of X. Then with the notation above,

- *either $i = dimE \leq dimE^s_k \oplus \cdots \oplus dimE^s_{a+1}$ (i.e. the dimension of E is smaller or equal than the dimension of the biggest stable escaping space).*

— *or $dim M - i - 1 = dim F \leq dim E^u_\ell \oplus \cdots \oplus dim E^u_{b+1}$ (i.e. the dimension of F is smaller or equal than the dimension of the biggest unstable escaping space).*

Proof. One argues by contradiction. One consider $L^s, L^u \in \mathbb{P}^c_{\sigma,U}$ so that L^s corresponds to a line in E^s_a and L^u a line in E^u_b. Assuming that the conclusion of the lemma is wrong, one gets that the projection of E^u_b on the normal space N_{L^s} is contained in $F(L^s)$ and the projection of E^s_a on the normal space N_{L^u} is contained in $E(L^u)$

There is $L \in \mathbb{P}^c_{\sigma,U}$, corresponding to a line in $E^s_a \oplus E^u_b$ and there are times r_n, s_n tending to $+\infty$ so that $L_{-n} = \phi_T^{-r_n}(L) \to L^s$ and $L_n = \phi_T^{s_n} \to L^u$.

Claim 40. *Given any $T > 0$ there is n and there are vectors u_n of the normal space $N_{L_{-n}}$ to L_{-n} so that the expansion of u_n by $\psi^T_{\mathcal{N}}$ is larger that $\frac{1}{2}$ times the minimum expansion in $F(L_{-n})$ and the contraction of the vector $\psi_T^{r_n+s_n}(u_n)$ by $\psi^T_{\mathcal{N}}$ is less than 2 times the maximal expansion in $E(L_n)$.*

The existence of such vectors u_n contradicts the domination $E \oplus_< F$ ending the proof. CQFD

According to Lemma 39 we now assume that $i \leq dim E^s_k + \oplus + dim E^s_{a+1}$ (the other case is analogous, changing X by $-X$).

Lemma 41. *With the hypothesis above, for every $L \in \mathbb{P}^c_{\sigma,U}$ the projection of $E^c_{\sigma,U}$ on the normal space $\mathcal{N}_L$ is contained in $F(L)$.*

Proof. It is because the projection of $E^s_k \oplus \cdots \oplus E^s_{a+1}$ has dimension at least the dimension i of E and hence contains $E(L)$. Thus the projection of $E^c_{\sigma,U}$ is transverse to E. As the projection of $E^s_{\sigma,U}$ on $\mathcal{N}_L$ defines a ψ^t_T-invariant bundle over the ϕ^t_T-invariant compact set $\mathbb{P}^c_{\sigma,U}$, one concludes that the projection is contained in F. CQFD

As a consequence the bundle F is not uniformly expanded on $\mathbb{P}^c_{\sigma,U}$ for the extended linear Poincaré flow. As it is expanded by the reparametrized flow, this implies $\sigma \in S_F$.

Consider now $L \in E^s_a$. Then $\psi^t_{\mathcal{N}}$ in restriction to the projection of $E^c_{\sigma,U}$ on $\mathcal{N}_L$ consists in multiplying the natural action of the derivative by the exponential contraction along L. As it is included in F, the multisingular hyperbolicity implies that it is a uniform expansion: this means that

— L is the unique contracting direction in $E^s_{\sigma,U}$: in other words, $a = 1$ and $dim E^s - a = 1$.

— the contraction along a is less than the expansion in the E^u_j, $j > 1$. In other words $E^c_{\sigma,U}$ is sectionally expanding.

For ending the proof of the Proposition 38, it remains to check the s-index of σ: at $L \in E^s_a$ one gets that $F(L)$ is isomorphic to $E^u_1 \oplus \cdots \oplus E^u_\ell$ so that the s-index of σ is $i + 1$, ending the proof.

4.4 Definition of singular volume partial hyperbolicity

Definition 26. Let X be a C^1-vector field on a compact manifold and let U be a compact region. We denote $S = Sing(X) \cap U$. One says that X is *singular volume partially hyperbolic* on U if

1. The restriction of the extended linear Poincaré flow $\{\psi_{\mathcal{N}}^t\}$ to the pre extended maximal invariant set $\widetilde{\Lambda}(X, U)$ admits a finest dominated splitting $\mathcal{N}_L = \mathcal{N}^s \oplus \cdots \oplus \mathcal{N}^u$ where $\mathcal{N}_L^s$ and $\mathcal{N}_L^u$ denote the extremal bundles.
2. There is a subset $S_{Ec} \subset S$ so that the reparametrized cocycle $h_{Ec}^t \psi_{\mathcal{N}}^t$ contracts volume in restriction to the bundles $\mathcal{N}_L^s$ over $\widetilde{\Lambda}(X, U)$ where h_{Ec} denotes
$$h_{Ec} = \Pi_{\sigma \in S_{Ec}} h_\sigma.$$
3. There is a subset $S_{Fc} \subset S$ so that the reparametrized cocycle $h_{Fc}^t \psi_{\mathcal{N}}^t$ expands volume in restriction to the bundle $\mathcal{N}_L^u$ over $\widetilde{\Lambda}(X, U)$ where h_{Fc} denotes
$$h_{Fc} = \Pi_{\sigma \in S_{Fc}} h_\sigma.$$

Remark 27. The subsets S_{Ec} and S_{Fc} are not necessarily uniquely defined, but they can be chosen to fit each context.

4.5 Extension of hyperbolicity along an orbit

Let us consider now a linear cocycle a linear cocycle $\mathcal{A}$ over (Λ, X), a hyperbolic set Λ for the cocycle $\mathcal{A}$ and an orbit y such that the α-limit of y, and the ω-limit of y are in Λ.

The splitting $E_{\alpha(y)} = E_{\alpha(y)}^s \oplus E_{\alpha(y)}^u$ will have an unstable cone field and a stable cone field that are strictly invariant on a neighborhood of Λ. Then the next lemma shows we can extend the hyperbolic structure of our cocycle to $\Lambda \cup o(y)$.

Lemma 42. *Let Λ be a hyperbolic, maximal invariant set in U, for $\mathcal{A}$, and $E_\Lambda = E^s \oplus E^u$. Suppose as well that*

— *The α-limit of y, $\alpha(y)$ is in Λ. Since Λ is hyperbolic then $E_{\alpha(y)} = E_{\alpha(y)}^s \oplus E_{\alpha(y)}^u$*

— *there exists a compact neighborhood U' such that $\Lambda \cup o(y)$ is a maximal invariant set in U',*

Then there exist a unique $E_y^u \subset E_y$ such that

— $dim(E_y^u) = dim(E_{\alpha(y)}^u)$

— *E_y^u is the set such that $E_y^u = \left\{ v \in E_y \text{ such that } \| \mathcal{A}^{-t}(v) \| \to 0 \right\}$*

— *This space is invariant, i.e. $\mathcal{A}^{-t}(E_y^u) = E_{\phi^{-t}(y)}^u$.*

— *The family of spaces* $\{ E^u(z) \}_{x\in\Lambda} \cup E^u_{\phi^{-t}(y)}$ *is continuous.*

Proof. Let us consider a smaller neighborhood U of Λ such that the family of unstable cones, given by the hyperbolicity of Λ, extends and is strictly invariant. Since the α limit of y is in Λ we can consider a time s such that $\phi^{-t}(y) \in U$ for every $s < t$. Let z be $\phi^{-s}(y) = z \in U$. there is a time t such that the unstable cone field $C^u(\phi^{-t}(z))$ around $\phi^{-t}(z)$ is mapped strictly inside the unstable cone field around z. Since $\phi^{-t}(z) \in U$ for every $0 < t$, we can take

$$F = \bigcap_{n\in\mathbb{N}} \mathcal{A}^{nt}\left(C^u(\phi^{-nt}(z))\right) .$$

The subspace of E_y given by $\mathcal{A}^s(F)$ is our desired space.

CQFD

Remark 28. If the ω-limit of y is in Λ we can have an analogous statement for the stable space.

We want to show that under similar assumptions the hyperbolicity of the reparametrized linear Poincaré flow of some maximal invariant set, extends to this set and one extra orbit with α and ω limits in this set. We now show that the extended maximal invariant set of $\Lambda \cup o(y)$ is the one of Λ and only one extra orbit. Then we will prove that the transversal intersection of the center stable and unstable spaces given by 42 which will give us the desired result.

Proposition 43. *Suppose that Λ is a multisingular hyperbolic, maximal invariant set in U. Suppose as well that*

— *y is such that the α and ω limits of y, $\alpha(y)$ and $\omega(y)$ are in Λ.*
— *there exists a compact neighborhood U' such that $\Lambda \cup o(y)$ is a maximal invariant set in U',*
— *The orbit of y does not intersect any escaping stable or unstable manifold of any singularity in Λ*

Then the extended maximal invariant set $\Lambda_{\mathbb{P}}(X, U')$ is $\Lambda_{\mathbb{P}}(X, U) \cup O(L)$ where $L = S_X(y)$ and $O(L)$ is the orbit of L by $\phi^t_{\mathbb{P}}$.

Proof. The set $S_X(\Lambda_{U'} \setminus Sing(X))$ gives only one point of $\mathbb{P}M$ for every regular point in the maximal invariant set of U'. Therefore

$$S_X(\Lambda_{U'} \setminus Sing(X)) = S_X(\Lambda_U \setminus Sing(X)) \cup O(L) .$$

The hypothesis above, that state that the orbit of y is away from the escaping stable and unstable manifolds of the singularity, and the fact that the α and ω limits of y are in Λ

$$S_X(\Lambda_U \setminus Sing(X)) \cup \overline{O(L)} \subset S_X(\Lambda_U \setminus Sing(X)) \cup O(L)\,.$$

Therefore

$$\begin{aligned}
\overline{S_X(\Lambda_{U'} \setminus Sing(X))} &= \overline{S_X(\Lambda_U \setminus Sing(X))} \cup \overline{O(L)} \\
&= \overline{S_X(\Lambda_U \setminus Sing(X))} \cup O(L) \\
&= \Lambda_{\mathbb{P}}(X,U) \cup O(L)\,.
\end{aligned}$$

CQFD

Corollary 44. *Suppose that Λ is a multisingular hyperbolic, maximal invariant set in U, for X.*

- *y such that the α and ω limits of y, $\alpha(y)$ and $\omega(y)$ are in Λ.*
- *there exists a compact neighborhood U' such that $\Lambda \cup o(y)$ is a maximal invariant set in U',*
- *The orbit of y does not intersect any escaping stable of unstable manifold of any singularity in Λ*
- *The stable and unstable spaces along the orbit of $S_X(y)$ given by Lemma 42 intersect transversally,*

Then $\Lambda \cup o(y)$ is multisingular hyperbolic.

Let us consider the set of chain recurrent points in a maximal invariant set $\Lambda \cap \mathcal{R}$ and suppose that this set is maximal invariant in a smaller neighborhood U', i.e.

$$\bigcap \phi^t(U') = \Lambda \cap \mathcal{R}\,.$$

Applying the same argument to a set of orbits in the hypothesis of proposition 43, we get that if the non chain recurrent orbits in a maximal invariant set do not intersect the escaping spaces of the singularities, then

$$B(X,U') \cup S(\Lambda \cap \mathcal{R}^c) = B(X,U)\,.$$

As a consequence:

Corollary 45. *Let Λ be the maximal invariant set in U. We consider the set of the chain recurrent orbits $\Lambda \cap \mathcal{R}$ and the set of the non chain recurrent orbits $\Lambda \cap \mathcal{R}^c$. We lift the chain recurrent orbits $S(\Lambda \cap \mathcal{R})$.If*

— *The set of chain recurrent orbits in the extended maximal invariant set* $B(X, U) \cap S(\Lambda \cap \mathcal{R})$ *is hyperbolic for the reparametrized linear Poincaré flow with the same index for all connected components).*

— *Every non chain recurrent orbit* $y \in \Lambda$ *does not intersect any escaping stable of unstable manifold of any singularity* Λ

— *The stable and unstable spaces along the lifted non chain recurrent orbit* $S_X(y)$ *given by Lemma 42 intersect transversally,*

Then Λ *is multisingular hyperbolic.*

Chapter 5

A star flow that is not singular hyperbolic in $\mathbb{R}^3$

This section will build of a chain recurrence class in M^3 containing two singularities of different indexes, that will be multisingular hyperbolic. However this will not be a robust class, and the singularities will not be robustly related. Other examples of this kind are exhibited in [BaMo]. A robust example is built in section 6 on a 5-dimensional manifold, since the results in [MPP] and [GLW] imply that in dimension 3 and 4 the star flows are, open and densely, singular hyperbolic in the usual sense (see 1.5.3). This structure forbids the coexistence of singularities of different indexes in the same class.

We add this example since it illustrates de situation in the simplest way we could.

Theorem 46. *There exists a vector field X in $S^2 \times S^1$ with an isolated chain recurrent class Λ such that :*

— *There are 2 singularities in Λ. They are Lorenz like and of different index.*
— *There is cycle between the singularities. The cycle and the singularities are the only orbits in Λ .*
— *The set Λ is multisingular hyperbolic.*

To begin with the proof of the theorem, let us start with the construction of a vector field X, that we will later show that it has the properties of the Theorem 46.

We consider a vector field in S^2 having:

— A source f_0 such that the basins of repulsion of f_0 is a disc bounded by a cycle Γ formed by the unstable manifold of a saddle s_0 and a sink σ_0.
— A source α_0 in the other component limited by Γ.

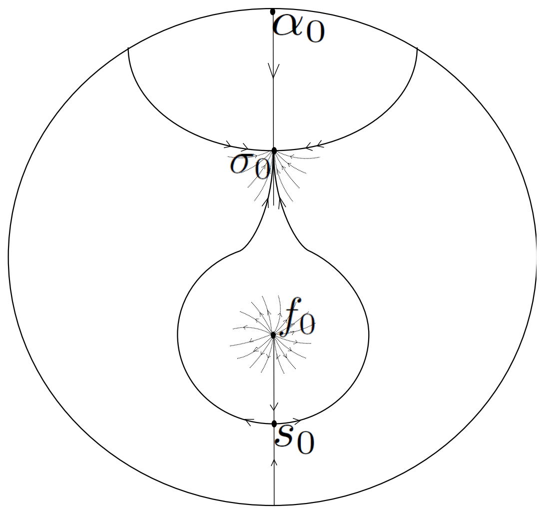


Figure 5.1 – The vector field in S^2

— We require that the tangent at σ_0 splits into 2 spaces, one having a stronger contraction than the other.

Note that the unstable manifold of s_0, is formed by two orbits. This two orbits have their ω-limit in σ_0, and as they approach σ_0, they become tangent to the weak stable direction, (see figure 5).

Now we consider S^2 embedded in S^3, and we define a vectorfield X_0 in S^3 that is normally hyperbolic at S^2, in fact we have S^2 times a strong contraction, and 2 extra sinks that we call ω_0 and P_0 completing the dynamics (see figure 5).

Note that σ_0 is now a saddle and the weaker contraction at σ_0 is weaker than the expansion. So σ_0 is Lorenz like.

Now we remove a neighborhood of f_0 and P_0. The resulting manifold is diffeomorphic to $S^2 \times [-1, 1]$ and the vector field X_0 will be entering at $A_0 = S^2 \times \{1\}$ and outing at $B_0 = S^2 \times \{-1\}$ (see figure 5).

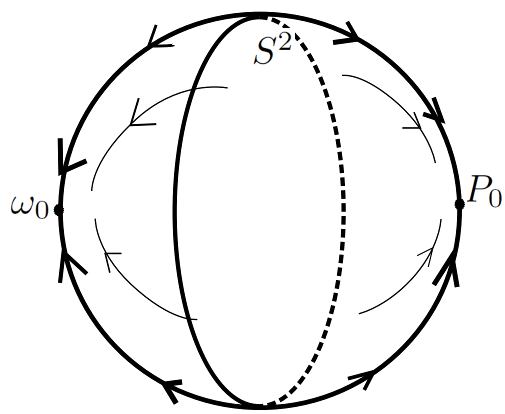


Figure 5.2 – S^2 normally repelling in S^3

Now we consider an other copy of $S^2 \times [-1, 1]$ with a vector field X_1 that is the reverse time of X_0. Therefore X_1 has a saddle called σ_1 that has a strong expansion, a weaker expansion and a contraction, and is Lorenz like. It also has a sink called α_1 a source called ω_1 and saddle called s_1.

The vector field X_1 is outing at $A_1 = S^2 \times \{1\}$ and entering at $B_1 = S^2 \times \{-1\}$.

We can now paste X_1 and X_0 along their boundaries (A_0 with A_1 and the other two). Since both vector fields are transversal to the boundaries we can obtain a C^1 vector field X in the resulting manifold that is diffeomorphic to $S^2 \times S^1$.

We do not paste B_0 with B_1 by the identity but with a rotation so that

$$\left(\overline{W^u(\alpha_0)} \cap B_0\right)^c$$

and

$$W^u(s_0) \cap B_0$$

are mapped to

$$W^s(\alpha_1) \cap (B_1) \ .$$

We require as well that

$$W^u(\sigma_0) \cap B_0$$

is mapped to

$$W^s(\sigma_1) \cap B_1 \,,$$

We will later require an extra condition on this gluing map, which is a generic condition, and

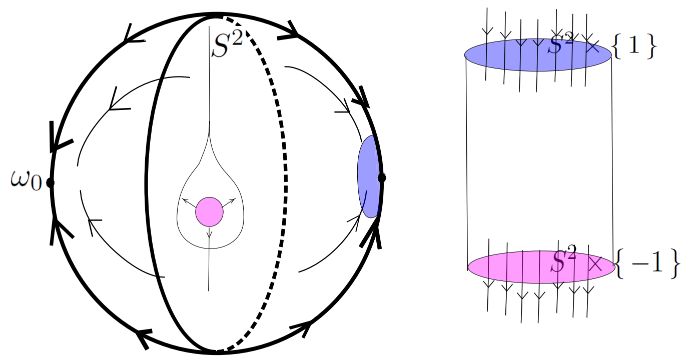


Figure 5.3 – Removing a neighborhood of f_0 and P_0

that will guarantee the multisingular hyperbolicity.

To glue A_0 with A_1, let us first observe that $W^s(\alpha_0) \cap A_0$ is a circle that we will call $\mathcal{C}_0$. We can also define the corresponding $\mathcal{C}_1$. We paste A_0 with A_1, mapping $\mathcal{C}_0$ to cut transversally $\mathcal{C}_1$.

Note that the resulting vector field X has a cycle between two Lorenz like singularities σ_0 and σ_1.

Lemma 47. *The vector field X defined above is such that the cycle and the singularities are the only chain recurrent points.*

Proof.
- All the recurrent orbits by X_0 in S^3 are the singularities. Once we remove the neighborhoods of the 2 singularities obtaining the manifold with boundary $S^2 \times [-1, 1]$, the only other orbits with hopes of being recurrent need to cut the boundaries.
- The points in B_0 that are not in $\overline{W^u(\alpha_0)}^c$ are not chain recurrent since they are mapped to the stable manifold of the sink α_1
- The points in $\overline{W^u(\alpha_0)}$ that are not in $W^u(\alpha_0)$ are in $W^u(s_0)$ or in $W^u(\sigma_0)$.
- The points in $B_0 \cap W^u(s_0)$ are mapped to the stable manifold of α_1.

As a conclusion, the only point in B_0 whose orbit could be recurrent is the one in

$$B_0 \cap W^u(\sigma_0)$$

. Let us now look at the points in A_0. There is a circle $\mathcal{C}_0$, corresponding to $W^s(\sigma_0) \cap A_0$ that divides A_0 in 2 components. One of this components is the basin of the sink ω_0 and the other

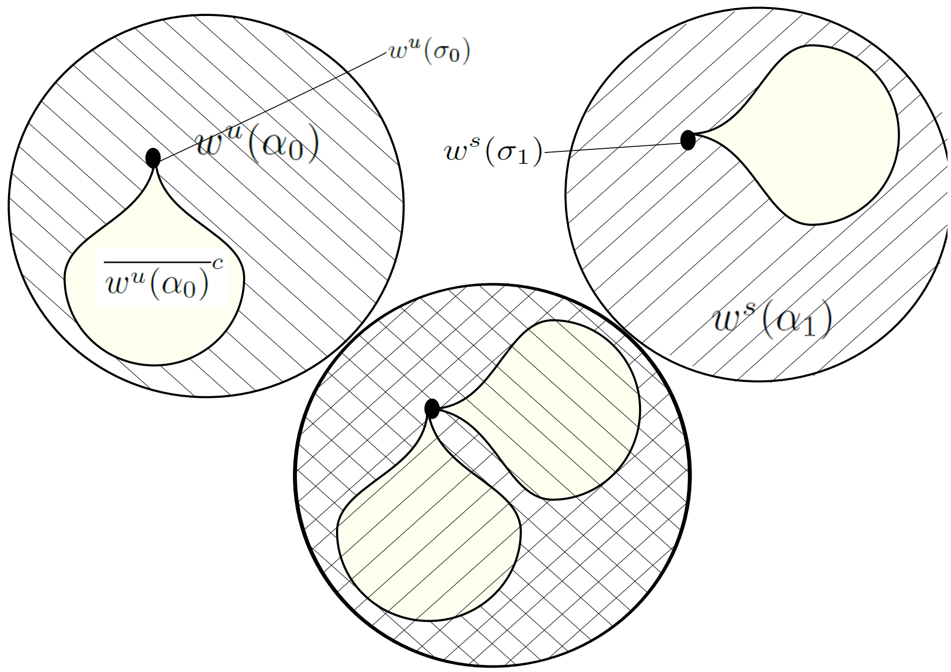


Figure 5.4 – Pasting $S^2 \times \{-1\}$ with $S^2 \times \{-1\}$.

is what used to be the basin of P_0. So we have the following options:

— The points that are in the basin of the sink ω_0 are not chain recurrent.

— The points that are in what used to be the basin of P_0 are either mapped into the basin of ω_1 or are sent to what used to be the basin of P_1. Note that this points cross B_0 for the past, and since they are not in the stable manifold of σ_1 they are not recurrent.

— Some points in $\mathcal{C}_0$ will be mapped to the basin of ω_1, others to what used to be the basin of P_1, and others to $\mathcal{C}_1$. In the two first cases those points are not recurrent.

To sum up,

— The only recurrent orbits that cross A_0, are in the intersection of $\mathcal{C}_0$ with$\mathcal{C}_1$.

— The only recurrent orbits that cross B_0, are in the intersection of $W^u(\sigma_0)$ with $W^s(\sigma_1)$.

— The only recurrent orbits that do not cross the boundaries of $S^2 \times [-1,1]$ are singularities.

This proves our lemma. CQFD

For the Lorenz singularity σ_0 of X which is of positive saddle value and such that $T_{\sigma_0}M = E^{ss} \oplus E^s \oplus E^{uu}$, we define $B_{\sigma_0} \subset \mathbb{P}M$ as

$$B_{\sigma_0} = \pi_{\mathbb{P}}\left(E^s \oplus E^{uu}\right) .$$

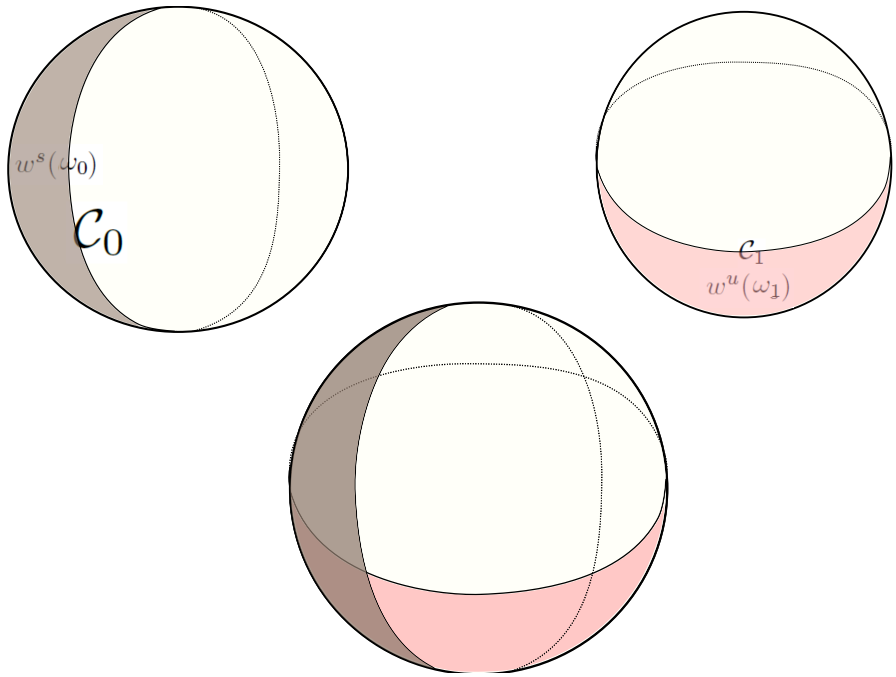


Figure 5.5 – Pasting $S^2 \times \{1\}$ with $S^2 \times \{1\}$.

For the Lorenz singularity σ_1 of X which is of positive saddle value and such that $T_{\sigma_1}M = E^{ss} \oplus E^u \oplus E^{uu}$, we define $B_{\sigma_1} \subset \mathbb{P}M$ as

$$B_{\sigma_1} = \pi_{\mathbb{P}}\left(E^{ss} \oplus E^u\right) .$$

Let a, b and c be points that are one in each of the 3 regular orbits forming the cycle between the two singularities of X. We call a to the one such that the α-limit of a is σ_0. We define $L_a = S_X(a)$ $L_b = S_X(b)$ and $L_c = S_X(c)$. We also note $O(L_a)$, $O(L_b)$ and $O(L_c)$ as the orbits of L_a, L_b and L_c by $\phi^t_{\mathbb{P}}$.

Proposition 48. *Suppose that X is a vector field defined above. Then there exist an open set U containing the orbits of a, b and c and the saddles σ_0 and σ_1, such that the extended maximal invariant set $B(U, X)$ is*

$$B(U, X) = B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \cup O(L_b) \cup O(L_c) .$$

Proof. The 2 orbits of strong stable manifold of σ_0 go by construction to α_0 for the past. This implies that the strong stable manifold is escaping. The fact that there is a cycle tells us that there are no other escaping directions, therefore the center space is formed by the weak stable and the unstable spaces. By definition $B_{\sigma_0} = \mathbb{P}^c_{\sigma_0}$. Analogously we see that $B_{\sigma_1} = \mathbb{P}^c_{\sigma_1}$.

Since the cycle formed by the orbits of a, b and c and the saddles σ_0 and σ_1 is an isolated chain recurrence class, we can chose U small enough so that this chain-class is the maximal invariant set in U. This proves our proposition. CQFD

Lemma 49. *We can choose a vector field X defined above is multisingular hyperbolic in U.*

Proof. The reparametrized linear Poincaré flow is Hyperbolic in restriction to the bundle over $B_{\sigma_0} \cup B_{\sigma_1}$ and of index one. We consider the set $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a)$.

The strong stable space at σ_0 is the stable space for the reparametrized linear Poincaré flow. There is a well defined stable space in the linearized neighborhood of σ_0 and since the stable space is invariant for the future, there is a one dimensional stable flag that extends along the orbit of a. We can reason analogously with the strong unstable manifold of σ_1 and conclude that there is an unstable flag extending through the orbit of a, and they intersect transversally.This is because this condition is open and dense in the possible gluing maps of $S^2 \times \{-1\}$ to $S^2 \times \{-1\}$, with the properties mentioned above. Therefore the set $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a)$ is hyperbolic for the reparametrized linear Poincaré flow.

Analogously we prove that $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \cup O(L_b) \cup O(L_c)$ is hyperbolic for the reparametrized linear Poincaré flow, and since from proposition 48 there exist a U such that,

$$B_{\sigma 1} \cup B_{\sigma 2} \cup O(L_a) \cup O(L_b) \cup O(L_c) = B(U, X) .$$

Then X is multisingular hyperbolic in U.

CQFD

The example in [BaMo] consists on two singular hyperbolic sets(negatively and positively) H_- and H_+ of different indexes, and wandering orbits going from one to the other. Since they are singular hyperbolic H_- and H_+ are multisingular hyperbolic sets of the same index. Moreover, the stable and unstable flags (for the reparametrized linear Poincare flow) along the orbits joining H_- and H_+ intersect transversally. This is also true for H_-.

With all this ingredients we can prove (in a similar way as we just did with the more simple example above) that the chain recurrence class containing H_- and H_+ in [BaMo] is multisingular hyperbolic, while it was shown by the authors that it is not singular hyperbolic.

Chapter 6

Robust example of a star flow that is not singular hyperbolic

In this Chapter we will prove Theorem 2. We recall it here

Theorem (j.w.Christian Bonatti)**.** *Let M be the manifold $S^3 \times \mathbb{RP}^2$. There is a C^1-open set $\mathcal{U}$ of $\mathcal{X}^1(M)$ so that every $X \in \mathcal{U}$*

- *is a star flow*
- *it has a chain class C having 2 singularities σ_1 and σ_2 such that the stable manifold of σ_1 is 3 dimensional and the stable manifold of σ_2 is 2 dimensional*
- *the singularities are such that σ_1 and σ_2 belong to $\overline{Per(X)}$*

We will first prove that there is a vector field X in R^3, that is star on a maximal invariant set in a neighborhood U, and that is not singular hyperbolic in U. This example is robust, but the maximal invariant set is not a chain recurrence class.

Theorem 50. *There exists an open set of vector fields $\mathcal{U} \subset \mathcal{X}^1(S^3)$ such that every $X \in \mathcal{U}$ has the following properties.*

- *There is a filtrating region $U = U_a \cap U_r$*
- *Λ is the maximal invariant set of a filtrating region U i.e.*

$$\Lambda = \bigcap_{t \in \mathbb{R}} \phi^t(U)$$

  where ϕ is the flow of X.
- *All singularities contained in Λ are strong Lorenz like.*
- *The set Λ contains a singularity σ_a that is accumulated by periodic orbits and that has a stable separatrix escaping U_a.*

— *Λ contains a singularity σ_r that is accumulated by periodic orbits and that has an unstable separatrix escaping U_r.*
— *There are orbits $o(y)$ in Λ such that the α-limit of y is in the chain-recurrent class of σ_r (that we call L_r) and the ω-limit of y is in the chain-recurrent class of σ_a (that we call L_a).*
— *The set Λ is multisingular hyperbolic.*

Remark 29. Notice that the previous theorem gives an open set of examples while in the previous section we presented a fragile example.

On a second step we will embed this example in M^5 and use the extra space to generate a chain recurrence class Λ_H with only one orbit $o(y)$ in Λ_H such that the ω-limit of y is in L_r and the α-limit of y is in L_a. This chain recurrence class will be multisingular hyperbolic.

Finally we perturb the chain class Λ_H to obtain more orbits going from L_a to L_r, in order to guarantee that the singularities will be robustly related. We use the fact that the multisingular hyperbolicity is open, to ensure this new vector field is still multisingular hyperbolic.

6.1 A multisingular hyperbolic set in $\mathbb{R}^3$

This section will be dedicated to the building a set in S^3 containing 2 singularities of different indexes that will be multisingular hyperbolic. However this set will not be recurrent.

Definition 30. We say that a hyperbolic singularity is strong Lorenz like if its tangent space splits into 3 invariant spaces. If the stable index is 2 then the Lyapunov exponents satisfy :

$$\lambda_a^{ss} < \lambda_a^s < 0 < -\lambda_a^s < \lambda_a^u < \lambda_a^{ss}.$$

If the unstable index is 2 then:

$$-\lambda_r^{uu} < \lambda_r^s < -\lambda_r^u < 0 < \lambda_r^u < \lambda_r^{uu}$$

Recall that this section is dedicated to prove:

Theorem. *There exists an open set of vector fields $\mathcal{U} \subset \mathcal{X}^1(S^3)$ such that every $X \in \mathcal{U}$ has the following properties.*
— *There is a filtrating region $U = U_a \cap U_r$*
— *Λ is the maximal invariant set of a filtrating region U i.e.*

$$\Lambda = \bigcap_{t \in \mathbb{R}} \phi^t(U)$$

where ϕ is the flow of X.

— *All singularities contained in Λ are strong Lorenz like.*
— *The set Λ contains a singularity σ_a that is accumulated by periodic orbits and that has a stable separatrix escaping U_a.*
— *Λ contains a singularity σ_r that is accumulated by periodic orbits and that has an unstable separatrix escaping U_r.*
— *There is an orbit $o(y)$ in Λ such that the α-limit of y is in the chain-recurrent class of σ_r (that we call L_r) and the ω-limit of y is in the chain-recurrent class of σ_a (that we call L_a).*
— *The set Λ is multisingular hyperbolic.*

6.1.1 The Lorenz attractor and the stable foliation

Is this subsection we will shortly comment on the construction of a geometric Lorenz attractor, done in [GuWi].

Guckenheimer Williams, geometric model

We consider a Flow in $\mathbb{R}^3$ as in [GuWi], having a transitive attractor with singularities that we call L_a. This set has the following properties:

— it has a singularity in the origin with three different real Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3$, with the following relation:

$$-\lambda_2 > \lambda_1 > -\lambda_3 > 0\,,$$

(We call this relation between the values *strong Lorenz like*, and it implies the Lorenz like condition)

— it is a robustly transitive, maximal invariant singular set, note that the singularity is accumulated by periodic orbits in a persistent way,
— there is an open region of attraction U_a in which L_a is the maximal invariant set. The boundary of this neighborhood is a torus $\mathbb{T}^2_a$.
— the strong stable spaces of the points in U_a is in the y direction.

Additionally the expansion rate is bounded form below by $\sqrt{2}$ and from above by 2 . This is a consequence of the way the example is constructed. So additionally we ask that the strong contraction rate is bigger that 4 and smaller than 5.

Attracting region:

Since we aim to construct an example in S^3, it will be more convenient to work with an attracting region U_a witch is a ball.

Let us consider two saddle singularities the holes of the toral trapping region from the above construction. This singularities will have 1 dimensional stable space and a 2 dimensional unstable space with complex Lyapunov exponents. The unstable spaces will cut the toral trapping region and the stable spaces are parallel to the y direction .

Then we can find an attracting region U_a such that the maximal invariant set contained on it is L^a and the 2 singularities, and the boundary of U_a (which is diffeomorphic to S^2) is S^2. For a more detail description we refer the reader to Guckenheimer Williams's work [GuWi].

We choose one of this 2 singularities p and we consider $C_a \subset U_a$, a subset having a smooth boundary homeomorphic to $D \times [0, \varepsilon]$ with axis one piece of the stable manifold of p. We ask that there exist $[\delta, \rho] \subset [0, \varepsilon]$ such that C_a and $D \times [0, \varepsilon]$ coincide exactly at $S^1 \times [\delta, \rho]$.

We ask that this cylinder C_a cuts the boundary of U_a and does not contain p.

Now we consider $U_a \setminus C_a$. The boundary of this new attracting region is such that the strong stable manifolds of the points in L_a, cut the boundary of the cylinder $D \times [\delta, \rho]$ parallel to the y direction, that is also parallel to the stable manifold of P. We can consider a function $h : U_a \setminus C_a \to U_a$ such that

— h is the identity except on a small neighborhood of the boundary (that doesn't intersect any recurrent orbit),

— h is a diffeomorphism.

— The image of restriction of h to $S^1 \times [\delta, \rho]$ is an annulus such that any line parallel to the axis goes to a radius. We call this annulus A_a

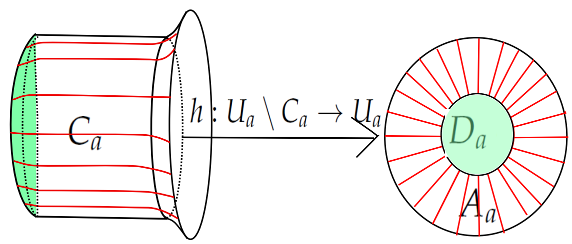
— Consider
$$C_a \setminus D \times [\delta, \rho] \,.$$
One of the connected components has a point of intersection of the stable manifold of P. We call the image of this component under h, D_a.

Finally we get an attracting region U_a such that :

— The boundary of U_a is S^2

— There is an annulus A_a in S^2 such that the strong stable manifolds of L_a intersect A_a along a radial foliation

— The annulus A_a bounds a disc D_a containing the intersection of the stable manifold of p and not of the other extra singularity.



6.1.2 The transversal intersection Tube

The goal of this section is to prove the following theorem:

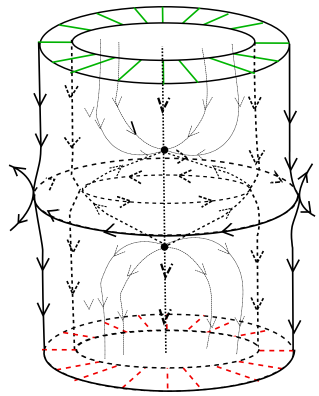
Theorem 51. *There exist a vector field χ such that its flow ϕ_χ defined in S^3has the following properties:*

There is a region $S^2 \times [0,1] \subset S^3$ such that

- *The vector field χ is enters $S^2 \times 0$ and points out in $S^2 \times 1$*
- *The vector field χ is such that the chain recurrent set consists of 2 sources singularities , p_1 and p_1', 2 sinks singularities, p_2 and p_2', and 2 periodic saddles, p_3 and p_3'.*
- *The intersection of the invariant manifolds of the saddles, with the boundary of $S^2 \times [0,1]$, are disjoint circles that we name as follows:*
  - *$W^s(p_3) \cap S^2 \times [0,1] = c_0$ in $S^2 \times 0$,*
  - *$W^s(p_3') \cap S^2 \times [0,1] = c_0'$ in $S^2 \times 0$,*
  - *$W^u(p_3) \cap S^2 \times [0,1] = c_1$ in $S^2 \times 1$,*
  - *$W^u(p_3') \cap S^2 \times [0,1] = c_1'$ in $S^2 \times 1$.*
- *The circle c_0 bounds a disc not containing c_0', that we call D_0. The circle c_0' bounds a disc containing c_0, that we call D_0'. And they both bound an annulus called A_0. Analogously we define D_1, D_1' and A_1.*
- *The orbit $O(x)$ of a point x in $S^2 \times \{0\}$, crosses $S^2 \times \{1\}$ if and only if $x \in A_0$ and $O(x) \cap S^2 \times \{1\} \in A_1$,*
- *There is a well defined crossing map $P : A_0 \to A_1$. We take polar coordinates in A_0 and A_1. Consider the radial foliation V_0 in A_0 such that the leaves are of the form $\theta \times [0,1]$ with $\theta \in S^1$. Then the image of V_0 under P intersect transversally a radial foliation V_1 in A_1 and it extends to a foliation in $A_1 \cup c_1 \cup c_1'$.*

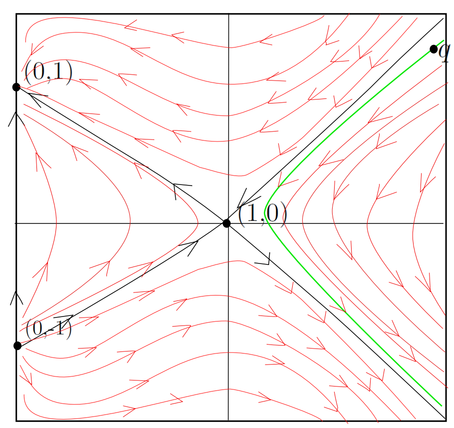
The complement of $S^2 \times [0,1]$ in S^3 are 2 balls, one in the basin of attraction of a source r (that has

$S^2 \times 0$ *in the boundary), and the other in the basin of attraction of a sink a.*



Most of the ideas here presented are similar than the ones in [BBY], the vector field that we aim to define is a plug in the sense of this article, and we refer the reader to this article to see a more careful presentation on how to glue plugs and what you can construct with them. We construct here a plug according to the specific needs of our example.

We consider the set $K = \{ (x, y) \ \ \text{tq} \ \ |y| \leq 1 \text{ and } 0 \leq x \leq 1 \}$, and in this set, a flow ϕ_0 of a vector field Y_0 in $\mathbb{R}^2$. The vector field Y_0 is Morse-Smale with a source $p_1 = (0, 1/2)$, a sink $p_2 = (0, -1/2)$ and a saddle $p_3 = (1/2, 0)$. We want the flow to be linear in a neighborhood of the interval $\{ (0, y) \ \ -1 \leq y \leq 1 \}$ for the saddle we want a branch of the unstable manifold to intersect the basin of the sink, and the other to intersect a corner of K. We want analogous properties for the stable manifold



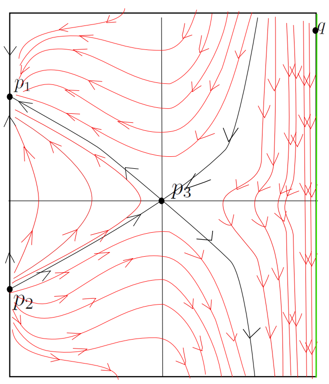
We take an orbit $q = (1, 1 - \epsilon)$ for some positive and small ϵ, that flows near the stable and unstable branch of the saddle that do not intersect the basins of the sink and the source. We consider another point of the orbit of q that we call q' with x coordinate 1. We choose ϵ so that all orbits of the points in the segment $\{(1,y) \ \ tq \ 1-\epsilon < y < 1\}$, cross the vertical segment that joints q' with $(1,-1)$. We call K' to the "square" delimited by

— the segments $\{(0,y) \ \ tq \ |y| \ \leq 1\}$,
— the segment $\{(x,1) \ \ tq \ 0 \leq x \leq 1\}$,
— the segments $\{(1,y) \ \ tq \ 1-\epsilon \leq y \leq 1\}$,
— the vertical segment that joints q' with $(1,-1)$,
— the orbit segment joining q and q',
— the segment $\{(x-,1) \ \ tq \ 0 \leq x \leq 1\}$.

We define

$$C = \{(x,y) \ \ | \ \ 0 \leq x \leq 2 \ \text{ y } \ |y| \ \leq 1\}\ .$$

There is a diffeomorphism $d : K' \to C$ that takes:



— the segments
  — $\{(x,1) \ \ tq \ 0 \leq x \leq 1\}$,
  — $\{(1,y) \ \ tq \ |y| \ \leq 1-\varepsilon\}$,
  to the segment $\{(x,1) \ \ tq \ 0 \leq x \leq 2\}$.
— the point q to $(2,1)$ and $(1,1)$ is fixed,
— the segments
  — $\{(x,1) \ \ tq \ 0 \leq x \leq 1\}$,
  — $\{(1,y) \ \ tq \ |y| \ \leq 1-\varepsilon\}$,
  to the segment $\{(x,1) \ \ tq \ 0 \leq x \leq 2\}$.
— the point q' to $(2,-1)$ and $(1,-1)$ is fixed

And so that d is the identity out of a small neighborhood of this boundary components. This neighborhood does not include any of the singular points.

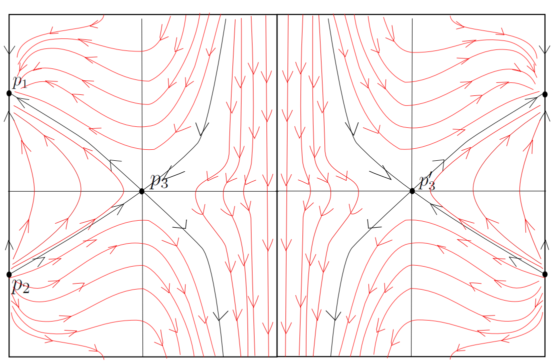
We call Y_1 to the vector field tangent to the flow ϕ_1 obtained from $d(\phi_0^t(d^{-1}(x)))$.
Now $(a, 0)$ are the new coordinates of the saddle p_3.
We define a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ such that:
— $f(0) = 0$
— $f(a) = 1$
— f is decreasing $(a, 2)$
— $f'(x) \neq 0$ in $[a, 2]$
— $f(2) = 0$.
Now we consider $C \times S^1$ and in S^1 we take the vector field

$$Y_2(\theta) = f(x)\, d\theta\,,$$

where $x \in [0, 2]$. We get the vector field $\chi^+ = (Y_1, Y_2)$ in the product. We can also consider

$$Y_3(\theta) = -f(x)\, d\theta\,,$$

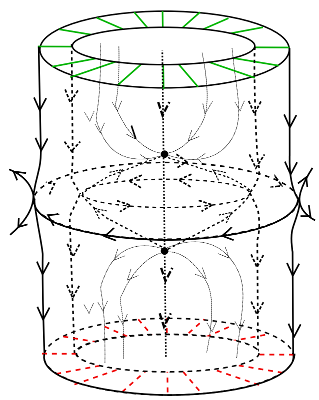
in another copy of $C \times S^1$. We get the vector field $\chi^- = (Y_1, Y_3)$ in the product. We call p'_1, p'_2 and p'_3 to the source, the sink and the saddle for χ^-.



We re write $C \times S^1$ as $\mathbb{D}^2 \times [-1, 1]$ and paste 2 copies of $\mathbb{D}^2 \times [-1, 1]$ along $\partial(\mathbb{D}^2) \times [-1, 1]$. In one copy we have χ^+ and in the other we have χ^-. Since the vector fields are equal in $\partial(\mathbb{D}^2) \times [-1, 1]$, both are C^∞ even restricted to the boundary and no orbit crosses $\partial(\mathbb{D}^2) \times [-1, 1]$, we get that the resulting vector field χ defined in $S^2 \times [-1, 1]$ is smooth.

The next lemma is to check all the conditions of theorem 52, except for the transversality condition, that we will check in the next subsection For our convenience, the intersections of the stable and unstable manifolds of the saddle periodic orbits are named as follows.

— $W^s(p_3) \cap S^2 \times [0, 1] = c_0$ in $S^2 \times 0$,



- $W^s(p'_3) \cap S^2 \times [0,1] = c'_0$ in $S^2 \times 0$,
- $W^u(p_3) \cap S^2 \times [0,1] = c_1$ in $S^2 \times 1$.
- $W^u(p'_3) \cap S^2 \times [0,1] = c'_1$ in $S^2 \times 1$.
- The intersection of the invariant manifolds of the saddles with the boundary of $S^2 \times [0,1]$ are the disjoint circles, c_0, c'_0, c_1, c'_1.
- The circle c_0 bounds a disc not containing c'_0, that we call D_0. The circle c'_0 bounds a disc not containing c_0, that we call D'_0. And they both bound an annulus called A_0 Analogously we define D_1, D'_1 and A_1.

We complete the vector-field to S^3 by adding 2 balls, one in the basin of attraction of a source r (that has $S^2 \times 0$ in the boundary), and the other in the basin of attraction of a sink a.

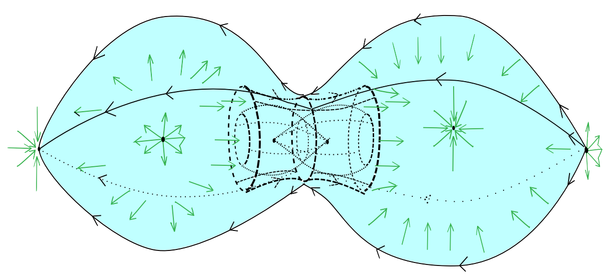


Figure 6.1 – The vector field χ in S^3.

Lemma 52. *The vector field χ defined in $S^2 \times [-1,1]$ has the following properties:*

— *The vector field χ is such that the chain recurrent set consists of 2 sources p_1 and p_1' , 2 sinks p_2 andp_2', and 2 periodic saddles p_3 and p_3'.*

— *The orbit $O(x)$ of a point x in a point in $S^2 \times 0$ crosses $S^2 \times 1$ if and only if $x \in A_0$ and $O(x) \cap S^2 \times 1 \in A_1$*

Proof. The flow ϕ_{Y_1} is such that the only recurrent points are the sinks or sources. This was not altered by the diffeomorphism d and by rotating it. By construction, there are no orbits crossing from one copy of $\mathbb{D}^2 \times [-1, 1]$ to the other. Also the intersection of the critical elements was transverse for ϕ_{Y_1} and this was also preserved.

The second item comes from the fact that all orbits of the points in the segment $\{ (1, y) \;\; tq \; 1 - \epsilon < y < 1 \}$, cross the vertical segment that joints q' with $(1, -1)$ for the flow ϕ_{Y_1}.

Then d takes this segments to $\{ (x, 1) \;\; tq \; 1 \leq x \leq 2 \}$ and $\{ (x, -1) \;\; tq \; 1 \leq x \leq 2 \}$. After symmetrizing we get that all orbits form $\{ (x, 1) \;\; tq \; 1 \leq x \leq 3 \}$ cross $\{ (x, -1) \;\; tq \; 1 \leq x \leq 3 \}$. Rotating this segments we obtain A_0 and A_1.

The converse also comes from the dynamics of ϕ_{Y_1}. The segment $\{ (x, 1) \;\; tq \; 0 \leq x < 1 \}$ is in the basin of attraction of the sink p_2. This remains true for Y_1, and therefore since D_0 is obtained by rotating this segment we get that this points are still in the basin of p_2 for χ. The same is true for the points in D_0' since the dynamic is analogous.

CQFD

6.1.3 A radial foliation and the image of the crossing map P

The aim of this subsection is to prove the last part of 52. Sice every orbit of the points in A_0 cuts A_1 at some moment, we define the first return map $P : A_0 \to A_1$. We take polar coordinates in A_0 and A_1 (that is, we take coordinates in $S^1 \times (0, 1)$). The diffeomorphism P can be written in this coordinates as

$$P(\theta, r) = (P_\theta(\theta, r), P_r(\theta, r))$$

Lemma 53. *Let $P : A_0 \to A_1$ be the first return map from A_0 cuts A_1 defined by the vector field χ. There is no value of $r \in (0, 1)$ for which the image under the differential of a vector tangent to the radial direction, has zero angular direction. As a consequence, the image of a radius under P cuts transversally any radius at A_1.*

Proof. Let us first consider the lift of A_0, $\widehat{A_0}$ which is a strip $\mathbb{R} \times (0, 1)$, we also take the lift of A_1, to $\widehat{A_1}$, and the lift of P, $\widehat{P}$. We orient this lifts by considering the rotation in the sense of p_3 as positive.

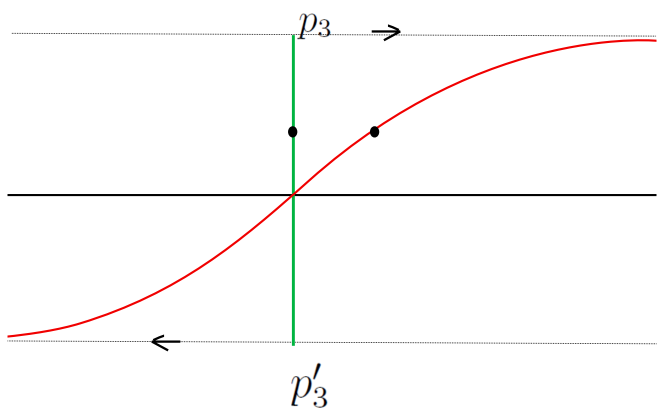


Figure 6.2 – The lift of the map $P : A_0 \to A_1$ to $\mathbb{R} \times (0, 1)$

Let us take a point $x = (\theta, r_x)$. The time that it takes for x to reach A_1 is T_x. We can extend the polar coordinates to the closure of A_0 and in this case $a_0 = S^1 \times \{0\}$.

Let us suppose that after the change of coordinates, the point previously in $x = 2$ is now $r = 1/2$. Let $g(r) = x$ be the change of coordinates. Suppose that $r < 1/2$ Recall that the vector field Y_2 is defined as

$$Y_2(\theta) = f(g(r))\, d\theta\,,$$

and therefore

$$\frac{\partial P_\theta(\theta, r)}{\partial r} = T_r f(g(r))' g(r)'\,,$$

and therefore non vanishing. Suppose that $r > 1/2$, then the vector field Y_3 is defined as

$$Y_3(\theta) = -f(g(r))\, d\theta\,,$$

and therefore

$$\frac{\partial P_\theta(\theta, r)}{\partial r} = T_r f(g(r))' g(r)'\,,$$

and therefore non vanishing.

At $r = 1/2$ since the lateral derivatives are not 0 and the function is smooth then

$$\frac{\partial P_\theta(\theta, r)}{\partial r} \neq 0\,.$$

CQFD

??

6.1.4 Gluing the pieces: defining a flow on S^3

Let us consider the vector field χ defined above, we remove the 2 balls in the complement $S^2 \times [0,1]$. We glue instead a ball which is the attracting region of a Lorenz attractor L_a and 2 singularities (the one from subsection 6.1.1), called U_a, instead of the ball that has $S^2 \times \{1\}$ on the boundary.

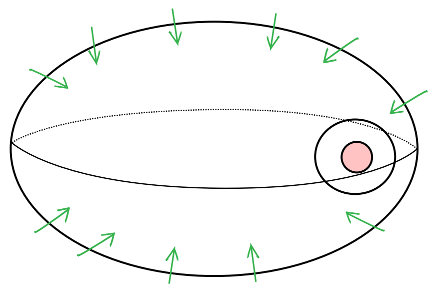


Figure 6.3 – The ball U_a.

Recall that from subsection 6.1.1 we have that

— The boundary of U_a is S^2

— There is an annulus A_a in S^2 such that the strong stable manifolds of L_a intersect A_a along a radial foliation

— The annulus A_a bounds a disc D_a containing the intersection of the stable manifold of p and not of the other extra singularity.

So we glue the boundary of U_a to $S^2 \times \{1\}$ so that

— A_a is mapped to an annulus containing A_0, a radial foliation of A_a is send to cut A_0 in a radial foliation.

— D_a is mapped inside D_0.

We consider a repelling region defined as the one from subsection 6.1.1, called U_r, but with the reverse time. The maximal invariant set in this ball is a Lorenz repeller L_r and 2 other singularities. We glue this ball instead of the ball that had $S^2 \times \{0\}$ in its boundary in an analogous way as we did with U_a.

Note that by doing this process we do not create any new recurrent orbits. We call the resulting vector field in S^3, X.

6.1.5 The filtrating neighborhood

Let us consider X from subsection 6.1.4.

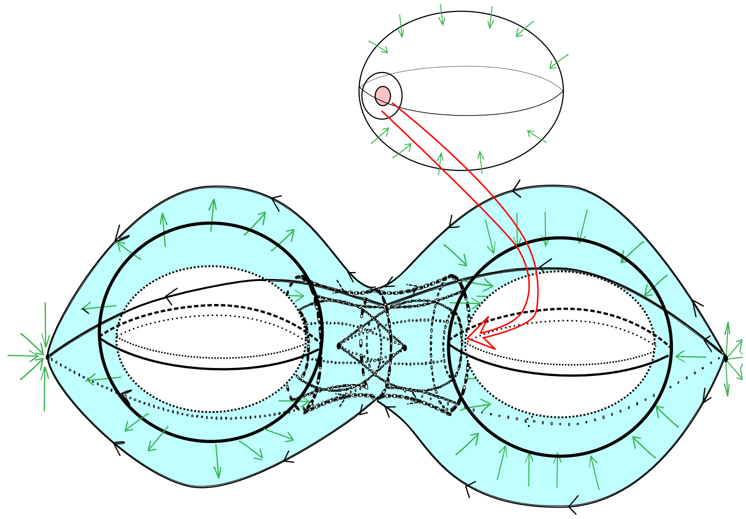


Figure 6.4 – Gluing U_a to $S^2 \times \{1\}$

If we remove some small neighborhoods inside the basin of the 2 sources p_1 and p_1' , we get a repelling region V_r.

If we remove some small neighborhoods inside the basin of the 2 sinks p_2 and p_2', we get an attracting region V_a.

The resulting open set

$$U = V_a \cap V_r$$

is a filtrating neighborhood. We call the maximal invariant set in it Λ.

Lemma 54. *For the vector field X the maximal invariant set $\Lambda \subset U$ is multisingular hyperbolic.*

Proof. The Lorenz attractor is singular hyperbolic, i.e.

$$T_x S^3 = E^{ss} \oplus E^{cu} \qquad \text{for all } x \in L_a$$

(see [MPP]). The strong stable space of L_a is escaping, and therefore the center space is E^{cu}. As a consequence

$$\begin{aligned} \mathcal{N}_L &= \pi_L(E^{ss} \oplus E^{cu}) \\ &= E^{ss} \oplus \pi_L(E^{cu}) \end{aligned}$$

is an invariant splitting of N_L for every $L \in B(X, U_a)$ The singularities in L_a are strong Lorenz like, and in fact, the expansion rate can never be bigger that 2 while the contraction rate is always bigger that 4. As a consequence

$$\Psi^t(L,u) = h(L,t) \cdot \psi^t_{\mathcal{N}}(L,u)$$

still contracts $\mathcal{N}^s(L) = E^{ss}$ since the biggest possible expansion rate for $h(L,t)$ is smaller than 2 . Since E^{cu} expands volume, that means that ,

$$\Psi^t(L,u) = h(L,t) \cdot \psi^t_{\mathcal{N}}(L,u)$$

expands $\mathcal{N}^u(L) = \pi_L(E^{cu}$. This proves at once that the splitting $\mathcal{N}_L = \mathcal{N}^s(L) \oplus \mathcal{N}^u(L)$ is dominated and that L_a is multisingular hyperbolic.

The periodic orbits are also multisingular hyperbolic since $h(L,t)$ does not expand or contract exponentially along a periodic orbit.

We need to check the the multisingular hyperbolicity in the non chain recurrent orbits that go from A_0 to A_1. For this, Lemma 45 tells us we need to check that the stable and unstable spaces that extend along this orbits, intersect transversely. This is a consequence of Lemma 53 and the fact that the stable foliation of L_a intersects A_1 radially, and the unstable foliation of L_r intersects A_0 radially. CQFD

6.2 A multisingular hyperbolic set in M^5

The objective of this section is to find a chain recurrent set that is multisingular hyperbolic with 2 singularities of different indexes. For this, the strategy will be to multiply the vector field X in S^3 from section 6.1 times a simple dynamic in $\mathbb{RP}^2$ and then modify the resulting set to obtain new recurrence.

The following lemma will be proven at the end of this section.

Lemma 55. *There exist a vector field Y in $\mathbb{RP}^2$ with the following properties:*

- *Y is a C^∞ vector field*
- *It has 3 singularities: a saddle singularity s, a source α and a sink ω and Y. It is linear in a neighborhood of the singularities.*
- *The contracting and expanding Lyapunov exponents of the saddle are equal in absolute value ($\lambda_{sss} = -\lambda_{uuu}$), and are very stronger that 6.*
- *One of the stable branches of s (that is an orbit) has its α-limit in α.*
- *One of the unstable branches of s (that is an orbit) has its ω-limit in ω.*
- *the other two branches form an orbit with α-limit and ω-limit in s and we call this orbit γ.*

— *There is a transverse section to γ and to the flow, that we call T to witch we can assign coordinates in $[-1,1]$. In this coordinates $T \cap \gamma = 0$ and the flow of Y, $\phi^Y(s,t)$ is such that:*
  — *If $s > 0$, then $\phi^Y(s,t)$ does not cross T for any $t > 0$ and has ω-limit in ω. And for $t < 0$ there exists only one $t_s < 0$ such that $\phi^Y(s,t_s) = s' \in T$ with $s' < 0$. and the α-limit of s is α.*
  — *If $s < 0$, then $\phi^Y(s,t)$ does not cross T for any $t < 0$ and has α-limit in α. And for $t > 0$ there exists only one $t_s > 0$ such that $\phi^Y(s,t_s) = s' \in T$ with $s' > 0$ and the ω-limit of s is ω.*

6.2.1 The vector field in M^5

We start by considering the vector field $Z_{id} = (X,Y)$ in the manifold $M^5 = S^3 \times \mathbb{RP}^2$ and it's flow ϕ_{id} . Let us define the section

$$\sum = S^3 \times T$$

which is transverse to Z_{id}, and a flow-box $\sum \times [-1,0]$.

Proposition 56. *Let $H : \sum \to \sum$ be a $\mathcal{C}^\infty$ diffeomorphism isotopic to identity and that is the identity on the boundary. There exist a $\mathcal{C}^1$ vector field Z_H such that $Z_H = Z_{id}$ in the complement of the flow-box $\sum \times [-1,0]$, and in the flow-box $(H(z),0) = Z_H((z,-1),1)$.*

Proof. Since H is isotopic to de identity we have that there exist a diffeomorphism $F : \sum \times [-1,0] \to \sum$ such that $F(\sum,-1) = id$ and $F(\sum,0) = H$. We also have that there exist $F' : \sum \times [-1,0] \to \sum$ such that $F'(\sum,-1) = H^{-1}$ and $F'(\sum,0) = id$. Let us define the flow ϕ_H as follows:

— $\phi_H(y,t) = \phi_{id}(y,t)$ for every t such that $\phi_H(y,t) \notin \sum \times [-1,0]$

— If t_0 is such that $\phi_H(y,t_0) \in \sum \times \{-1\}$ then

$$\phi_H(y,t) = F(\phi_{id}(y,t_0),s)\,,$$

for every $s = t - 1 - t_0$ such that $-1 \leq s \leq 0$.

— If t_1 is such that $\phi_H(y,t_1) \in \sum \times \{0\}$ then

$$\phi_H(y,t) = F'(\phi_{id}(y,t_1),s)\,,$$

for every $t = s - 1 - t_1$ such that $-1 \leq s \leq 0$.

Now we define the vector field Z_H by taking at any point, the derivative (on t) of $\phi_H(y,t)$ and since $\phi_H(y,t)$ is sufficiently smooth, then so is Z_H. CQFD

A filtrating region for Z_H

Now in a similar way we consider all sources for X in S^3 and we consider a repelling region that we call $v_f \subset S^3$. We also consider a trapping region v_p of all the sinks in S^3.

We recall that U is a filtrating region defined in Section 6.1. we consider $U' = U/(u_p \cup u_f)$. We define now the filtrating region in M^5 that is interesting to us:

$$V = U_0 \cap (U' \times \mathbb{RP}^2)\,.$$

Proposition 57. *The maximal invariant set Λ_{id} in V (for Z_{id}) intersects $\sum$. And for any H as above, any orbit in the maximal invariant set $\Lambda_H \in V$ (for Z_H) either crosses $\sum$ or is contained in $S^3 \times \{ s \}$.*

Proof. Let us consider the saddle singularity in Y that we called s. By construction, there is a unique orbit of Y, formed by a branch of the stable and unstable manifold of s, that crosses T. Since the contraction and expansion rates in Y are stronger than in X, then the points in $S^3 \times \{ s \}$ have a connection between the strong stable and unstable manifolds and the orbits in this connections cross $\sum$.

If the orbit γ_y of a point $y = \{ (x, l) \}$ never crosses $\sum$ then

$$Z_{id} \mid_{\gamma_y} = Z_H \mid_{\gamma_y}\,.$$

Let us see that Λ_{id} is contained in $S^3 \times \{ s \}$ or it crosses $\sum$. We take $u_0 = \mathbb{RP}^2/u_\alpha \cup u_\omega$. Then the maximal invariant set in u_0 for Y is the saddle s and the saddle connection (the orbit that contains one unstable branch and one stable branch of s). All other points have their α and ω-limits in the singularities α and ω (see the properties of Y in (55)). So if there is a point $y \in \gamma_y$ such that $y \notin S^3 \times \{ s \}$ and $\gamma_y \cap \sum = \emptyset$ then the orbit of l by Y has α or ω-limits in the singularities α and ω. This implies that y has α and ω-limits in $U_\alpha \cup U_\omega$. Therefore $\gamma_y \notin \Lambda_{id}$. CQFD

We consider a repelling region $u_\alpha \subset \mathbb{RP}^2$ of α for Y, such that α is the maximal invariant set in u_α. Similarly, consider a trapping region $u_\omega \subset \mathbb{RP}^2$ We take the respective repelling and trapping regions of this singularities in M^5. We define the repelling region $U_\alpha = S^3 \times u_\alpha$ and the trapping region $U_\omega = S^3 \times u_\omega$. We define as well

$$U_0 = M^5 / \{ U_\alpha \cup U_\omega \}$$

.

We recall that there are 2 saddles singularities in S^3, σ_a and σ_r. By construction of the Lorenz attractor (see [GuWi]) there is a small linear neighborhood around the singularity, in which we can consider the coordinates (x, y, z) to correspond to the strong unstable, weak stable and stable spaces. The singularity is approached by orbits of L_a only in one semispace that corresponds to the points with positive y value. We say then that σ_a has an escaping separatrix W^{cs-} which is the half stable manifold that escapes from a neighborhood of L_a. In the same way there is an escaping separatrix W^{cu+} for the singularity σ_r in L_r.

We consider a small neighborhood

$$u_a = \{ (x, y, z) \} \quad \text{such that} \quad -\delta < x < \delta \quad -\delta < z < \delta \quad -\delta < y < 0$$

choosin δ so that u_a is in the linearized neighborhood of σ_a.

Analogously we define u_r for σ_r. Note that here the stable and unstable manifolds refer to de dynamics of X. We define now the corresponding repelling and trapping regions in M^5. That is $V_i = \mathbb{RP}^2 \times v_i$ for $i = \{ r, a \}$.

6.2.2 The chain recurrent set with different singularities

We are going to start with a flow Z_{id} which is a skew product and alter some cross section of it by a diffeomorphism H so that the result is a multisingular set in M^5. For that we now need to choose some more properties on the diffeomorphism H from proposition 57. The following lemma will be proven in section 6.4.

Lemma 58. *There exist a C^∞ diffeomorphism isotopic to identity, $H : \Sigma \to \Sigma$, where $\sum = S^3 \times T$, that is the identity on the boundary. We take coordinates for T in $[-1, 1]$ and in this coordinates,*

$$H(x, l) = (r_l(x), \theta_x(l))$$

where $r_l : \sum \to S^3$, $\theta_x : \sum \to T$. We can construct such a function having the following properties:

- *The map $r_l(x)$ is the identity for $l = 1$ or $l = -1$, or if $x \in u_a \cup u_r$.*
- *Consider a compact ball $B_r \subset S^3$ that intersects the maximal invariant set $\Lambda \in U$ only in a point $z' \in W^s_X(\sigma_r)$. Analogously consider a compact ball B_a that intersects the maximal invariant set $\Lambda \in U$ only in a point $z \in W^u_X(\sigma_a)$.*
  The image of $r_l(B_a) = B_r$, and $r_l(z) = z'$ for all $l \in [-1/2, 1/2]$.
- *The balls B_a and B_r can be taken so that there exist $K_Y > t_0$ such that $\phi^t_X(B_r) \subset (u_r)$ and $\phi^{-t}_H(B_a) \subset (u_a)$ for all $t > t_0$. Recall that $K_Y + 1$ is the minimum of the times that it takes for a point in T_1 to return to T for Y and $K_Y > 0$.*
- *If $l \in [-1/2, 1/2]^c$ then $\theta_x(l) = l$.*
- *If $l \in [0, 1/2]$ and $x \notin B_a$ $\theta_x(l) > 0$.*
- *If $l \geq 0$ and $x \in B_a$ then $-\epsilon < \theta_x(l) \leq \epsilon$,*

— *The only point l such that $H(z,l) = (z',0)$, is $l = 0$.*

Proposition 59. *We consider H as before, then the orbits in the maximal invariant set Λ_H are contained in $\Lambda \times \{ s \}$ or cross the flow box $\sum \times [-1,0]$ in*

$$B_a \times [0, 1/2] \times \{ -1 \} .$$

Proof. Suppose that γ is an orbit in Λ_H that doesn't cross $\sum$. From Proposition 57 these orbits of Λ_H are in $S^3 \times \{ s \}$. Let y be a point of γ of coordinates (x, p). If x is not in Λ (for X) then, the alpha or the omega limit of x must be in U'^c. Therefore, for a t large enough,

$$\phi_H^t(x,l) \notin V = U_0 \cap U' .$$

Then if γ doesn't intersect $\sum$, it must be in $\Lambda \times \{ s \}$.

Let us suppose now that γ intersects $\sum$. Let y be a point in $\gamma \cap \sum \times [-1,0]$ such that $y \in \sum \times \{ -1 \}$. We write y as $(z, -1)$ and z as $z = (x,l) \in \sum$.

1. If $l > 1/2$, or if $x \notin B_a$ with $l > 0$, then $H(x,l) = (r_l(x), \theta(l))$ with $\theta(l) > 0$ and then $\phi_H^1(y) = (r_l(x), \theta(l)) \times \{ 0 \}$. Since outside of the flow-box $Z_{id} = Z_H$ now we can look at Z_{id}. From the properties of Y (55) we have that the future orbit of $\theta(l) > 0$, does not cross T and the ω-limit is ω. Then the orbit for ϕ_H^t is in U_ω for a large enough t. Then γ is not in Λ_H.
2. If $l < 0$, since y goes outside of the flow-box for the past (where $Z_{id} = Z_H$) now we can look at Z_{id}. From the properties of Y (55) we have that the orbit of $l < 0$ does not cross T for the past and the α-limit is α. Then γ does not cross again the flow-box for the past. The orbit for ϕ_H^t is in U_α for a negatively large enough t and γ is not in Λ_H.
3. If x is not in B_a and $l = 0$ then $\phi_H^1(y) = ((H(x), \theta_x(l)), 0)$ and $\theta_x(l) > 0$. Then, as before, we have that the orbit of $\theta_x(l)$ for Y does not cross T for the future and the ω-limit for Y is ω. Then γ does not cross again the flow-box for the future and γ is not in Λ_H.

Then the only other case in which $\gamma \in \Lambda_H$ is if γ crosses the flow box $\sum \times [-1,0]$ in $B_a \times [0, 1/2] \times \{ -1 \}$. CQFD

Proposition 60. *There is a unique orbit γ in Λ_H that crosses $\sum$, that orbit is the orbit of*

$$(z, 0) \times \{ -1 \} \in \sum .$$

Proof. Let γ be an orbit in Λ_H. From proposition (59), we already know that if an orbit of Λ_H crosses $\sum$ then it crosses at a point $y = (x, l, -1) \in B_a \times [0, 1/2] \times \{ -1 \}$.

If $\theta_x(l) < -\epsilon < 0$ recall that the properties of H (Lemma 58) give us that then $l < 0$.

Suppose now that $l \geq 0$ and that $\theta_x(l) > 0$. As in our previous proposition this implies that $\phi_H^{t+1}(y) \notin U$ for t large enough.

If $l \geq 0$ and $-\epsilon < \theta_x(l) \leq 0$ then $x \in B_a$. Suppose that $x \neq z$. Then

$$\phi_H^{t+1}(y) \notin \sum \times [-1, 0]$$

for all $K_Y > t > 0$, and therefore $\phi_H^{t+1}(y) = \phi_{id}^t(\phi_H^1(y))$ for all $K_Y > t > 0$. Let us consider t_0 as in the properties of H (Lemma 58). Recall that t_0 is such that $\phi_X^t(B_r) \subset (v_r)$ and $\phi_H^{-t}(B_a) \subset (v_a)$ for all $K_Y > t > t_0$. We call $\phi_{id}^{t_0}(\phi_H^1(y)) = (x_1, z_1) \in S^3 \times \mathbb{RP}^2$. Since $x_1 \in u_r$ where u_r is an attracting region for D_r and such that

$$\bigcap_{t \in \mathbb{R}^+} \phi_X^t(u_r) = p_r$$

where p_r is a sink of X (see subsection 6.1.4). Now, for all $t > t_0$ even the ones bigger than K_Y,if we call $s = t_0 + 1 - t$, we have that

$$\phi_H^s(x_1, z_1) = (\phi_{Hx}^s(x_1, z_1), \phi_{Hz}^s(x_1, z_1))$$

and since every time for the future that this orbit crosses the flow-box $\sum \times [-1, 0]$, the function r_l is the identity, then

$$\phi_H^s(x_1, z_1) = (\phi_X^s(x_1), \phi_{Hz}^s(x_1, z_1))\,.$$

Since $\phi_X^t(x_1) \notin U$ for $t > t_0$ big enough, then $\phi_H^{t'}(y)$ is eventually not in V for some $t > t_0$. Then γ is not in Λ_H as wanted.

If $l \geq 0$ and $-\epsilon < \theta_x(l) < 0$ but $x = z$. Let t_y be a time in which the orbit returns to the flow-box. That is t_y is such that

$$\phi_H^{t_y}(y) = (x_1, l_1, -1) \in \sum \times \{\,-1\,\}\,.$$

Recall from the properties of H (Lemma 58) that $t_y \geq K_Y > t_0$ with t_0 such that $\phi_X^t(B_r) \subset (u_r)$ and $\phi_H^{-t}(B_a) \subset (u_a)$ for all $t > t_0$. Since$-\epsilon < \theta_x(l) \leq 0$ and after returning to $\sum \times \{\,-1\,\}$ the orientation was reversed, then l_1 is positive. Since now x_1 is not in B_a, then $\theta_{x_1}(l_1) > 0$. So, now for any $t > 0$, we have that

$$\phi_H^{t+t_y+1}(y) = \phi_{id}^t(\phi_H^{t_y+1}(y))\,.$$

This implies that the orbit of y never cuts the flow box again, and therefore, for a big enough t, $\phi^t_H(y)$ is in U_ω. As a consequence γ is not in Λ_H as wanted.

The only case left is $x = z$ and $\theta_x(l) = 0$. The last property of H (Lemma 58) tells us that $l = 0$, so the objective now is to prove that the orbit of $y = (z, 0)$ never leaves

$$V = U_0 \times U' \times \mathbb{RP}^2 .$$

But $(z, 0)$ is in the stable manifold of σ_r and in the unstable manifold of σ_a, and then then the orbit of y is in Λ_H..

CQFD

6.2.3 Multisingular hyperbolicity

Until now we have constructed a vector field having a chain recurrent class such that

— Two singularities of different indexes one in L_a and the other in L_r.

— All the periodic orbits have the same index and the singularities are in the closure of the periodic orbits.

— There are periodic orbits in L_a such that their stable manifolds intersect the unstable manifolds of periodic orbits in L_r.

— There is only one orbit in the class with the α-limit in L_a and the ω-limit in L_r.

The goal now is to show that we can choose a diffeomorphism H so that this vector field would be multisingular hyperbolic. After that we will perturb this vector field to an other that will still be multisingular hyperbolic, but having a homoclinic connection between periodic orbits in L_a and periodic orbits in L_r. This will finish the proof of theorem 2.

In the following section we will prove not only that there exist a diffeomorphism H the properties defined in Lemma 58, but also that this function can be constructed with the following additional property, we require that the image of

$$S^3 \times \{0\} \times \{-1\} \in \Sigma \times \{-1\}$$

under H cuts transversally

$$S^3 \times \{0\} \times \{0\} \in \Sigma \times \{0\} .$$

This last property guaranties that the set Λ_H will be multisingular hyperbolic.

To show that Λ_H is multisingular hyperbolic we need to check that we are in the hypothesis of 44 Since we have already shown the other The following lemma implies the multisingular hyperbolicity

Lemma 61. *Let $y \in \Sigma \times \{-1\}$ be such that $\Lambda_H = \Lambda_\cup O(y)$. There exist a diffeomorphism H such that*

— *The stable and unstable spaces along the orbit of $S_X(y)$ intersect transversally,*

— *The orbit of y does not intersect the escaping spaces of the singularities for Z_H,*

then Λ_H is multisingular hyperbolic.

Proof. Consider the points $a = (z, s)$, $b = (z', s)$ and $y = (z, 0, -1) \in \Sigma$. The orbit of y is in the strong unstable manifold on a, (since unstable manifold of s intersects T at 0 for Y). Analogously y is in the strong stable manifold on b since $\phi_H^1(y) = (z', 0, 0)$. Observe that a and b are regular orbits and $a \in W^u(\sigma_a)$ and $b' \in W^s(\sigma_r)$ for X. therefore γ does not intersect the escaping spaces of the singularities for Z_H. From Proposition 43 this implies that the center space of the singularities of Λ_H and Λ are the same.

From Lemma 42 we have that there exist an unstable space (for the reparametrized linear Poincaré flow) at a that we call E_y^u. We chose a metric so that the normal space at y is tangent to $\Sigma \times \{-1\}$. We take a vector $v \in E_y^u$ at y. This vector is tangent to

$$S^3 \times \{0\} \times \{-1\} \in \Sigma \times \{-1\}$$

at y. Let us recall that we have assumed at the beginning of the subsection that the image of

$$S^3 \times \{0\} \times \{-1\} \in \Sigma \times -1$$

under H cuts transversally

$$S^3 \times \{0\} \times \{0\} \in \Sigma \times \{0\} .$$

Then the image of v under the differential of H (and of ϕ_H^1) is transverse to $S^3 \times 0 \times O(y)$ at $\phi_H^1(y)$, and then so is the image of v under $\Psi^1(v)$, since the direction of the flow is not tangent to $T \times \{0\}$. On the other hand Lemma 42 also gives us a stable space E_y^s at $\phi_H^1(y)$ that is tangent to $S^3 \times \{0\} \times \{0\}$ at $\phi_H^1(y)$. Then the stable and unstable spaces of the reparametrized linear Poincaré flow are transversal. Then we are in the hypothesis of 44 and this completes the proof. CQFD

With this last lemma we know that the maximal invariant set Λ_H is multisingular hyperbolic. But this is not enough, since a small perturbation of Z_H could brake the connection between L_a and L_r and have σ_a and σ_r in different chain classes. We need now to show that the right perturbation of Z_H will generate the intersection of the stable and unstable manifolds of periodic orbits in L_a and L_r. Since Λ_H is multisingular hyperbolic for Z_H, so will it be for this new vector field and now the singularities will be robustly in the same chain recurrence class.

The following lemma implies Theorem 2

Lemma 62. *There is an arbitrarily small perturbation of Z_H, that we call Z_{H_ϵ}, and a C^1 neighborhood of Z_{H_ϵ} called $\mathcal{V}$ so that any vector field $Z \in \mathcal{V}$ has a maximal invariant set Λ_Z that is multisingular hyperbolic and there is a chain class $C \in \Lambda_Z$ that has two singularities of different index.*

Proof. We will make a small perturbation of H and this will result in a small perturbation of Z_H. Let us recall that we can write H as

$$H(x,l) = (r_l(x), \theta_x(l))$$

where $r_l : \sum \to S^3$, $\theta_x : \sum \to T$.

We will only perturb $r_l(x)$ to $r_l'(x)$ so that $\circ B_a \cap L_a = b_a$ is a small ball (relative to L_a), and the same for b_r. we can also ask that $r_l'(b_a) \cap b_r$. This can be done with an arbitrarily small C^r perturbation, so that the resulting vector field Z_{H_ϵ} is still C^1 and multisingular hyperbolic.

Note that since b_a and b_r are open, then there is a small neighborhood of $r_l'(x)$ and therefore a small neighborhood of Z_{H_ϵ} , $\mathcal{V}$ so that the image of b_a intersects b_r for all vector fields in the neighborhood.

Now from the fact that periodic orbits are dense in the sets L_a and L_r, and the fact that Z_{H_ϵ} is star , we get that we can choose a small perturbation by 17 so that the unstable manifold of some periodic p orbit in L_a intersects transversally the stable manifold of a periodic orbit q in L_r. Recall that the periodic orbits all have the same index.

Also by the connecting lemma we can get by an other small enough perturbation, that the stable manifold of p intersects the stable manifold of q. This homoclinic intersection is roust. CQFD

6.3 Construction of the vector field Y in $\mathbb{RP}^2$

6.3.1 A vector field with a saddle connection in a Möbius strip

Let us start by defining some simple linear flow in $\mathbb{R}^2$. We take a linear vector field $Y(x,y) = (\lambda_{sss}x, \lambda_{uuu}y)$ defined in $[-2,2] \times [-2,2]$. We ask that $\lambda_{uuu} = -\lambda_{sss}$ and we also ask that $\lambda_{uuu} > 6$.

We consider a close curve C formed by the union of following curves:

— We consider the orbit of a point $(-a,2)$. This orbit cuts the vertical line $(-2,y)$ in a point $(-2,a')$. The segment of orbit from $(-2,a')$ to $(-a,2)$ is our first curve C_1.

— We consider the orbit of a point $(a, 2)$. This orbit cuts the vertical line $(2, y)$ in a point $(2, c)$. The segment of orbit from $(a, 2)$ to $(2, c)$ is C_2.
— We consider the segment $\{-2\} \times [a', -a']$ as our second curve C_3.
— We take the orbit of $(-2, -a')$ and we call the point where it cuts the horizontal line l in a point $(-b, -2)$. The segment of orbit from $(-2, -a')$ to $(-b, -2)$ is our third curve C_4.
— We consider the segment $\{2\} \times [-c, c]$ as our second curve C_5.
— We consider the orbit of a point $(2, -c)$. This orbit cuts the horizontal line $(x, -2)$ in a point $(b', -2)$. The segment of orbit from $(2, -c)$ to $(b', -2)$ is C_6.
— The segment $[b', -b] \times \{-2\}$ our forth curve C_7.
— The segment $[-a, a] \times \{2\}$ our last curve C_8.

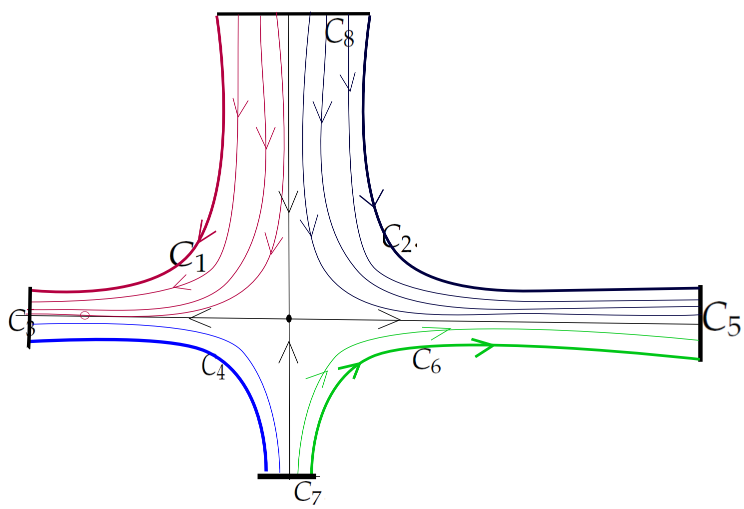


Figure 6.5 – The vector field Y in D.

There is a diffeomorphism $d : C_8 \to C_5$ defined as follows:

$$d(x) = -\frac{c(x)}{a}.$$

Now we glue C_8 and C_5 along d. There is a connected component in the complement of C that contains $(0, 0)$. We call the closure of this connected component D. The manifold D (with boundary C) obtained from this gluing is a 2 dimensional non-orientable manifold with a connected boundary, therefore it is a Möbius strip.

Note that since the $d : C_8 \to C_5$ is such that 0 is pasted to 0, then there is a branch of the stable manifold of $(0,0)$ and a branch of the unstable manifold of $(0,0)$ that intersect. That is, there is an orbit γ such that

$$\gamma \subset W^s(0,0) \cap W^u(0,0)\,.$$

We say then that $(0,0)$ has a saddle connection.

6.3.2 Completing the vector field to $\mathbb{RP}^2$

Let us consider a linear vector field in $\mathbb{R}^2$ with a sink ω, and let us take a neighborhood u_ω in its basin of attraction. We choose a curve in the boundary, it will be pointing inwards. We can take C_3, and since the vector field Y is pointing outwards , we can paste them.

Note that the remanning unstable branch has its ω-limit in ω.

We call the new vector field Y and what remains of the boundary of u_ω, we now call it C_3'.

Analogously we attach a neighborhood u_α, containing a source α and glue it through the segment C_7. We call the subset of boundary of u_α, that was not glued to D, C_7'.

Note that the remaining stable brunches of $(0,0)$ (that is an orbit) hast its α-limit in α.

We call D' to the region formed by D with u_α and u_ω attached. Since D' is a Möbius strip, then the complement in $\mathbb{RP}^2$ is a disc R having a boundary formed by 4 disjoint curves tangent to the flow (C_1, C_2, C_6 and C^4), one curve transverse to the flow and entering D' C_7', and one curve transverse to the flow and exiting D' C_3'. Therefore we can define the flow in the complement of D' in the trivial way by sending the points in C_3' to C_7'.

Now we prove Lemma 55

Proof. — Since the original maps are linear, the resulting map after the gluing is also $\mathcal{C}^\infty$.

- — The contracting and expanding Lyapunov values of Y can be taken to be as strong as required
- — As noted above, one branch of each stable and unstable manifold form a saddle connection γ while the others come or go to the sink and source.
- — The segment, $T_0 = C_8$ is a transverse section to γ by construction and is such that:
  - — If $s > 0$ $\phi^Y(s,t)$ never touches T_0 for any $t > 0$ and has ω-limit in ω. And for $t < 0$ there exists only one $t_s < 0$ such that $\phi^Y(s,t_s) = s' \in T_0$ with $s' < 0$. and the α-limit of s is α.
  - — If $s < 0$ $\phi^Y(s,t)$ never touches T_0 for any $t < 0$ and has α-limit in α. And for $t > 0$ there exists only one $t_s > 0$ such that $\phi^Y(s,t_s) = s' \in T_0$ with $s' > 0$ and the ω-limit

of s is ω.

As a consequence of the fact that that C_8 was glued to C_5 reverting orientation.

CQFD

6.4 Construction of the diffeomorphism H

In this section we prove the following lemma from the previous section:

Lemma. *(58) There exist a $\mathcal{C}^\infty$ diffeomorphism isotopic to identity, $H : \Sigma \to \Sigma$ that is the identity on the boundary. We consider $\sum = S^3 \times T$ and we take coordinates for T in $[-1, 1]$,*

$$H(x, l) = (r_l(x), \theta_x(l))$$

where $r_l : \sum \to S^3$, $\theta_x : \sum \to T$. We can construct such a function having the following properties:

- *The map $r_l(x)$ is the identity for $l = 1$ or $l = -1$, or if $x \in u_a \cup u_r$.*
- *Consider a compact ball $B_r \subset S^3$ that intersects the maximal invariant set $\Lambda \in U$ only in a point $z' \in W^s_X(\sigma_r)$. Analogously consider a compact ball B_a that intersects the maximal invariant set $\Lambda \in U$ only in a point $z \in W^u_X(\sigma_a)$. The image of $r_l(B_a) = B_r$, and $r_l(z) = z'$ for all $l \in [-1/2, 1/2]$.*
- *There exits $K_Y > t_0$ such that $\phi^t_X(B_r) \subset (u_r)$ and $\phi^{-t}_H(B_a) \subset (u_a)$ for all $t > t_0$. Recall that $K_Y + 1$ is the minimum of the times that it takes for a Point in T_1 to return to T for Y and $K_Y > 0$.*
- *If $l \in [-1/2, 1/2]^c$ then $\theta_x(l) = l$.*
- *If $l \in [0, 1/2]$ and $x \notin B_a$ $\theta_x(l) > 0$.*
- *If $l \geq 0$ and $x \in B_a$ then $-\epsilon \leq \theta_x(l) \leq \epsilon$,*
- *The only point l such that $H(z, l) = (z', 0)$, is $l = 0$.*
- *The image of*

  $$S^3 \times \{0\} \times \{-1\} \in \Sigma \times -1$$

  under H cuts transversally

  $$S^3 \times \{0\} \times \{0\} \in \Sigma \times \{0\} .$$

Proof. Let us consider a closed neighborhood of $v_a \cup v_r$ that we call C. Since B_a and B_r are subsets of S^3, they are isotopic to each other. Moreover, (choosing C correctly) they are isotopic to each other in $S^3 \setminus C$, since $v_a \cup v_r$ does not disconnect S^3. Therefore there is a function $r' : S^3 \times [0,1/2] \to S^3$ such that

$$r'(x,0) = id(x) \text{ and } r(B_a, 1/2) = B_r.$$

We can choose r' so that it is the identity in the boundary of C for all $t \in [0,1]$ and such that $r'(z,1/2) = z'$ We can extend now this function to S^3 by asking that $r' \mid v_a \cup v_r = Id$. Now $r : S^3 \times [-1,1] \to S^3$ is defined by

$$r(x,l) = \begin{cases} r'(x, l+1), & \text{if} l \leq -1/2 \\ r'(x, 1/2), & \text{if if } -1/2 < l \leq 1/2 \\ r'(x, l-1), & \text{if} l > 1/2 \,. \end{cases} \tag{6.1}$$

Now we need to construct $\theta : \sum \to [-1,1]$.

We consider a $\mathcal{C}^\infty$ bump function $h : [-1,1] \to [-1,1]$,

— If $l \in [-1/2, 1/2]^c$ then $h(l) = 0$,

— if $l \in [-1/2, 1/2]$ then $0 < h(l) \leq \frac{\epsilon}{2}$,

— if $l = 0$ then $h(l) = \frac{\epsilon}{4}$,

— $\frac{\partial h(l)}{\partial (l)} \mid_{(0)} \neq 0$

We can also assume that h is sufficiently differentiable. Let B_A be an arbitrarily small neighborhood of B_a. We consider now a second bump function $g : S^3 \to [-1,1]$

— If $x \in B_A^c$ then $g(x) = 0$,

— If $x \in B_a$ then $\epsilon \leq g(x) \leq \frac{\epsilon}{2}$,

— $g(z) = -\frac{\epsilon}{4}$ and $\frac{g(x)}{\partial (v)} \mid_{(z)} \neq 0$ for any given v direction in S^3.

We define then θ_x as follows:

$$\theta_x(l) = id(l) + h(l) + g(x) \,.$$

Note that the image of the vectors tangent to the coordinates in S^3, under the differential of H, have a non vanishing component in the direction of T. This is our desired function. CQFD

Chapter 7

The multisingular hyperbolicity is a necessary condition for star flows

The aim of this section is to prove lemma 63 below:

Lemma 63 (j.w with Christian Bonatti)**.** *Let X be a generic star vector field on M. Consider a chain-recurrent class C of X. Then there is a filtrating neighborhood U of C so that the extended maximal invariant set $B(X,U)$ is multisingular hyperbolic.*

Notice that, as the multi singularity of $B(X,U)$ is a robust property, lemma 63 implies theorem 6.

As already mentioned, the proof of lemma 63 consists essentially in recovering the results in [GSW] and adjusting few of them to the new setting. So we start by recalling several of the results from or used in [GSW].

To start we state the following properties of star flows:

Lemma 64 ([L][Ma2])**.** *For any star vector field X on a closed manifold M, there is a $\mathcal{C}^1$ neighborhood $\mathcal{U}$ of X and numbers $\eta > 0$ and $T > 0$ such that, for any periodic orbit γ of a vector field $Y \in \mathcal{U}$ and any integer $m > 0$, let $N = N_s \oplus N_u$ be the stable- unstable splitting of the normal bundle N for the linear Poincaré flow ψ_t^Y then:*

— Domination: *For every $x \in \gamma$ and $t \geq T$, one has*

$$\frac{\| \psi_t^Y |_{N_s} \|}{\min(\psi_t^Y |_{N_u})} \leq e^{-2\eta t}$$

— Uniform hyperbolicity at the period: *at the period If the period $\pi(\gamma)$ is larger than T then, for every $x \in \gamma$, one has:*

$$\Pi_{i=0}^{(m\pi(\gamma)/T)-1} \qquad \| \psi_t^Y |_{N_s} (\phi_{iT}^Y(x)) \| \leq e^{-m\eta\pi(\gamma)}$$

and

$$\Pi_{i=0}^{(m\pi(\gamma)/T)-1} \qquad \min(\psi_t^Y |_{N_u} (\phi_{iT}^Y(x))) \geq e^{m\eta\pi(\gamma)}.$$

Here $\min(A)$ *is the mini-norm of* A*, i.e.,* $\min(A) = \| A^{-1} \|^{-1}$.

Now we need some generic properties for flows:

Lemma 65 ([C][BGY])**.** *There is a* C^1*-dense* G_δ *subset* $\mathcal{G}$ *in the* C^1*-open set of star flows of* M *such that, for every* $X \in \mathcal{G}$*, one has:*

- *Every critical element (singularity or periodic orbit) of* X *is hyperbolic and therefore admits a well defined continuation in a* C^1*-neighborhood of* X*.*
- *For every critical element* p *of* X*, the Chain Recurrent Class* $C(p)$ *is continuous at* X *in the Hausdorff topology;*
- *If* p *and* q *are two critical elements of* X*, such that* $C(p) = C(q)$ *then there is a* C^1 *neighborhood* $\mathcal{U}$ *of* X *such that the chain recurrent class of* p *and* q *still coincide for every* $Y \in \mathcal{U}$
- *For any nontrivial chain recurrent class* C *of* X*, there exists a sequence of periodic orbits* Q_n *such that* Q_n *tends to* C *in the Hausdorff topology.*

Lemma 66 (Lemma 4.2 in [GSW])**.** *Let* X *be a star flow in* M *and* $\sigma \in Sing(X)$*. Assume that the Lyapunov exponents of* $\phi_t(\sigma)$ *are*

$$\lambda_1 \leq \cdots \leq \lambda_{s-1} \leq \lambda_s < 0 < \lambda_{s+1} \leq \lambda_{s+1} \leq \cdots \leq \lambda_d$$

. If the chain recurrence class$C(\sigma)$ *of* σ*, is nontrivial, then:*

- *either* $\lambda_{s-1} \neq \lambda_s$ *or* $\lambda_{s+1} \neq \lambda_{s+2}$*.*
- *if* $\lambda_{s-1} = \lambda_s$*, then* $\lambda_s + \lambda_{s+1} < 0$*.*
- *if* $\lambda_{s+1} = \lambda_{s+2}$*,* $\lambda_s + \lambda_{s+1} > 0$*.*
- *if* $\lambda_{s-1} \neq \lambda_s$ *and* $\lambda_{s+1} \neq \lambda_{s+2}$*, then* $\lambda_s + \lambda_{s+1} \neq 0$*.*

We say that a singularity σ in the conditions of the previous lema is *Lorenz like* of index s and we define the *saddle value* of a singularity as the value

$$sv(\sigma) = \lambda_s + \lambda_{s+1}.$$

Consider a Lorenz like singularity σ, then:

— if $sv(\sigma) > 0$, we consider the splitting

$$T_\sigma M = G^{ss}_\sigma \oplus G^{cu}_\sigma$$

where (using the notations of Lemma 66) the space G^{ss}_σ corresponds to the Lyapunov exponents λ_1 to λ_{s-1}, and G^{cu}_σ corresponds to the Lyapunov exponents $\lambda_s, \dots, \lambda_d$.

— if $sv(\sigma) < 0$, we consider the splitting

$$T_\sigma M = G^{cs}_\sigma \oplus G^{uu}_\sigma$$

where the space G^{cs}_σ corresponds to the Lyapunov exponents λ_1 to λ_{s+1}, and G^{uu}_σ corresponds to the Lyapunov exponents $\lambda_{s+2}, \dots, \lambda_d$.

Corollary 67. *Let X be a vector field and σ be a Lorenz-like singularity of X and let $h_\sigma \colon \Lambda_X \times \mathbb{R} \to (0, +\infty)$ be a cocycle in the cohomology class $[h_\sigma]$ defined in Section 4.1.*

1. *First assume that $Ind(\sigma) = s+1$ and $sv(\sigma) > 0$. Then the restriction of $\psi_\mathcal{N}$ over $\mathbb{P}G^{cu}_\sigma$ admits a dominated splitting $\mathcal{N}_L = E_L \oplus F_L$, with $dim(E_L) = s$, for $L \in \mathbb{P}G^{cu}_\sigma$. Furthermore,*
   — *E is uniformly contracting for $\psi_\mathcal{N}$*
   — *F is uniformly expanding for the reparametrized extended linear Poincaré flow $h_\sigma \cdot \psi_\mathcal{N}$.*
2. *Assume now that $Ind(\sigma) = s$ and $sv(\sigma) < 0$. One gets a dominated splitting $\mathcal{N}_L = E_L \oplus F_L$ for $\phi_\mathcal{N}$ for $L \in \mathbb{P}G^{cs}_\sigma$ so that $dim(E_L) = s$, the bundle F is uniformly expanded under $\psi_\mathcal{N}$ and E is uniformly contracted by $h_\sigma \cdot \psi_\mathcal{N}$.*

Proof. We only consider the first case $Ind(\sigma) = s+1$ and $sv(\sigma) > 0$, the other is analogous and can be deduced by reversing the time.

We consider the restriction of $\psi_\mathcal{N}$ over $\mathbb{P}G^{cu}_\sigma$, that is, for point $L \in \tilde{\Lambda}_X$ corresponding to lines contained in G^{cu}_σ. Therefore the normal space $\mathcal{N}_L$ can be identified, up to a projection which is uniformly bounded, to the direct sum of G^{ss}_σ with the normal space of L in G^{cu}_σ.

Now we fix $E_L = G^{ss}_\sigma$ and F_L is the normal space of L in G^{cu}_σ. As G^{ss}_σ and G^{cu}_σ are invariant under the derivative of the flow ϕ_t, one gets that the splitting $\mathcal{N}_L = E_L \oplus F_L$ is invariant under the extended linear Poincaré flow over $\mathbb{P}G^{cu}_\sigma$.

Let us first prove that this splitting is dominated:

By Lemma 66 if we choose a unit vector v in E_L we know that for any $t > 0$ one has

$$\| \psi^t_\mathcal{N}(v) \| \leq K e^{t\lambda_{s-1}}.$$

Now let us choose a unit vector u in F_L, and consider $w_t = \psi^t_\mathcal{N}(u) \in F_{\phi^t_\mathbb{P}(L)}$. Then for any

$t > 0$, one has

$$\| D\phi^{-t}(w_t) \| \leq K' e^{t(-\lambda_s)} \| w_t \| .$$

The extended linear Poincaré flow is obtained by projecting the image by the derivative of the flow on the normal bundle. Since the projection on the normal space does not increase the norm of the vectors, one gets

$$\| \psi_{\mathcal{N}}^{-t}(w_t) \| \leq K' e^{t(-\lambda_s)} \| w_t \| ,$$

is other words

$$\frac{1}{\| \psi_{\mathcal{N}}^{t}(u) \|} \leq K' e^{t(-\lambda_s)}$$

Putting together these inequalities one gets:

$$\frac{\| \psi_{\mathcal{N}}^{t}(v) \|}{\| \psi_{\mathcal{N}}^{t}(u) \|} \leq KK' e^{t(\lambda_{s-1}-\lambda_s)} .$$

This provides the domination as $\lambda_{s-1} - \lambda_s < 0$.

Notice that $E_L = G_\sigma^{ss}$ is uniformly contracted by the extended linear Poincaré flow $\psi_{\mathcal{N}}$, because it coincides, on G_σ^{ss} and for $L \in \mathbb{P}G_\sigma^{cu}$, with the differential of the flow ϕ^t. For concluding the proof, it remains to show that the reparametrized extended linear Poincaré flow $h_\sigma \cdot \psi_{\mathcal{N}}$ expands uniformly the vectors in F_L, for $L \in \mathbb{P}G_\sigma^{cu}$.

Notice that, over the whole projective space $\mathbb{P}_\sigma$, the cocycle $h_{\sigma,t}(L)$ is the rate of expasion of the derivative of ϕ_t in the direction of L. Therefore $h_\sigma \cdot \psi_{\mathcal{N}}$ is defined as follows: consider a line $D \subset \mathcal{N}_L$. Then the expansion rate of the restriction of $h_\sigma \cdot \psi_{\mathcal{N}}$ to D is the expansion rate of the area on the plane spanned by L and D by the derivative of ϕ_t.

The hypothesis $\lambda^s + \lambda^{s+1} > 0$ implies that the derivative of ϕ_t expands uniformly the area on the planes contained in G_σ^{cu}, concluding.

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Lemma 68 (Lemma 4.5 and Theorem 5.7 in [GSW]). *Let X be a C^1 generic star vector field and let $\sigma \in Sing(X)$. Then there is a filtrating neighborhood U of $C(\sigma)$ so that, for every two periodic points $p, q \subset U$,*

$$Ind(p) = Ind(q),$$

Furthermore, for any singularity σ' in U,

$$Ind(\sigma') = Ind(q) \qquad \text{if } sv(\sigma) < 0\,,$$

or

$$Ind(\sigma') = Ind(q) + 1 \qquad \text{if } sv(\sigma) > 0\,.$$

Lemma 69. *There is a dense G_δ set $\mathcal{G}$ in the set of star flows of M with the following properties: Let X be in $\mathcal{G}$, let C be a chain recurrent class of X. Then there is a (small) filtrating neighborhood U of C so that the lifted maximal invariant set $\widetilde{\Lambda}(X,U)$ of X in U has a dominated splitting $\mathcal{N} = E \oplus_{\prec} F$ for the extended linear Poincaré flow, so that E extends the stable bundle for every periodic orbit γ contained in U.*

Proof. According to Lemma 68, the class C admits a filtrating neighborhood U in which the periodic orbits are hyperbolic and with the same index. On the other hand, according to Lemma 65, every chain recurrence class in U is accumulated by periodic orbits. Since X is a star flow, Lemma 64 asserts that the normal bundle over the union of these periodic orbits admits a dominated splitting for the linear Poincaré flow, corresponding to their stable/unstable splitting. It follows that the union of the corresponding orbits in the lifted maximal invariant set have a dominated splitting for $\mathcal{N}$. Since any dominated splitting defined on an invariant set extends to the closure of this set, we have a dominated splitting on the closure of the lifted periodic orbits, and hence on the whole $\widetilde{\Lambda}(X,U)$. CQFD

Lemma 69 asserts that the lifted maximal invariant set $\tilde{\Lambda}(X,U)$ admits a dominated splitting. What we need now is extend this dominated splitting to the extended maximal invariant set

$$B(X,U) = \tilde{\Lambda}(X,U) \cup \bigcup_{\sigma_i \in Sing(X)\cap U} \mathbb{P}^c_{\sigma_i,U}.$$

Now we need the following theorem to have more information on the projective center spaces $\mathbb{P}^c_{\sigma_i,U}$.

Lemma 70 (Lemma 4.7 in [GSW]). *Let X be a star flow in M and σ be a singularity of X such that $C(\sigma)$ is nontrivial. Then :*

— *if $sv(\sigma) > 0$, one has:*

$$W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}\,,$$

where $W^{ss}(\sigma)$ is the strong stable manifold associated to the space G^{ss}_σ.

— *if $sv(\sigma) < 0$, then:*

$$W^{uu}(\sigma) \cap C(\sigma) = \{\sigma\}\,,$$

where $W^{uu}(\sigma)$ is the strong unstable manifold associated to the space G^{uu}_σ.

Remark 31. Consider a vector field X and a hyperbolic singularity σ of X. Assume that $W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}$, for a strong stable manifold $W^{ss}(\sigma)$, where $C(\sigma)$ is the chain recurrence class of σ.

Then there is a filtrating neighborhood U of $C(\sigma)$ on which the strong stable manifold $W^{ss}(\sigma)$ is escaping from U (see the definition in Section 3.2).

Proof. Each orbit in $W^{ss}(\sigma) \setminus \{\sigma\}$ goes out some filtrating neighborhood of $C(\sigma)$ and the nearby orbits go out of the same filtrating neighborhood. Notice that the space of orbits in $W^{ss}(\sigma) \setminus \{\sigma\}$ is compact, so that we can consider a finite cover of it by open sets for which the corresponding orbits go out a same filtrating neighborhood of $C(\sigma)$. The announced filtrating neighborhood is the intersection of these finitely many filtrating neighborhoods. CQFD

Remark 31 allows us to consider the *escaping strong stable* and *strong unstable manifold* of a singularity σ without refereing to a specific filtrating neighborhood U of the class $C(\sigma)$: these notions do not depend on U small enough. Thus the notion of the *center space* $E^c_\sigma = E^c(\sigma, U)$ is also independent of U for U small enough. Thus we will denote

$$\mathbb{P}^c_\sigma = \mathbb{P}^c_{\sigma,U}$$

for U sufficiently small neighborhood of the chain recurrence class $C(\sigma)$.

Remark 32. Lemma 70 together with Remark 31 implies that:

— if $sv(\sigma) > 0$, then the center space E^c_σ is contained in G^{cu}

— if $sv(\sigma) < 0$, then $E^c_\sigma \subset G^{cs}$.

Lemma 71. *Let X be a generic star vector field on M. Consider a chain recurrent class C of X. Then there is a neighborhood U of C so that the extended maximal invariant set $B(X, U)$ has a dominated splitting for the extended linear Poincaré flow*

$$\mathcal{N}_{B(X,U)} = E \oplus_{\prec} F$$

which extends the stable-unstable bundle defined on the lifted maximal invariant set $\widetilde{\Lambda}(X, U)$.

Proof. The case where C is not singular is already done. According to Lemma 68 there an integer s and a neigborhood U of C so that every periodic orbit in U has index s and every singular point σ in U is Lorenz like, furthermore either its index is s and $sv(\sigma) < 0$ or σ has index $s + 1$ and $sv(\sigma) > 0$.

According to Remark 32, one has:

$$B(X,U) \subset \tilde{\Lambda}(X,U) \cup \bigcup_{sv(\sigma_i)<0} \mathbb{P}G^{cs}_{\sigma_i} \cup \bigcup_{sv(\sigma_i)>0} \mathbb{P}G^{cu}_{\sigma_i}$$

By Corollary 67 and Lemma 69 each of this set admits a dominated splitting $E \oplus F$ for the extended linear Poincaré flow $\psi_{\mathcal{N}}$ with $dimE = s$.

The uniqueness of the dominated splittings for prescribed dimensions implies that these dominated splitting coincides on the intersections concluding. CQFD

We already proved the existence of a dominated splitting $E \oplus F$, with $dim(E) = s$, for the extended linear Poincaré flow over $B(X,U)$ for a small filtrating neighborhood of C, where s is the index of any periodic orbit in U. It remains to show that the extended linear Poincaré flow admits a reparametrization which contracts uniformly the bundle E and a reparametrization which expands the bundle F.

Lemma 66 divides the set of singularities in 2 kinds of singularities, the ones with positive saddle value and the ones with negative saddle value. We denote

$$S_E := \{x \in Sing(X) \cap U \text{ such that } sv(x) < 0\} \quad \text{and,}$$

$$S_F := \{x \in Sing(X) \cap U \text{ such that } sv(x) > 0\}.$$

Recall that Section 4.1 associated a cocycle $h_\sigma\colon \Lambda_X \to \mathbb{R}$, whose cohomology class is well defined, to every singular point σ.

Now we define

$$h_E = \Pi_{\sigma \in S_E} h_\sigma \text{ and } h_F = \Pi_{\sigma \in S_F} h_\sigma.$$

Now Lemma 63 and therefore Theorem 6 are a direct consequence of the next lemma:

Lemma 72. *Let X be a generic star vector field on M. Consider a chain recurrent class C of X. Then there is a neighborhood U of C so that the extended maximal invariant set $B(X,U)$ is such that the normal space has a dominated splitting $\mathcal{N}_{B(X,U)} = E \oplus_{\prec} F$ such that the space E (resp. F) is uniformly contracting (resp. expanding) for the reparametrized extended linear Poincaré flow $h^t_E \cdot \psi^t_{\mathcal{N}}$ (resp. $h^t_F \cdot \psi^t_{\mathcal{N}}$).*

The proof uses the following theorem by Gan Shi and Wen, which describes the ergodic measures for a star flow. Given a C^1 vector field X, an ergodic measure μ for the flow ϕ_t, is said to be *hyperbolic* if either μ is supported on a hyperbolic singularity or μ has exactly one zero Lyapunov exponent, whose invariant subspace is spanned by X.

Theorem 73 (lemma 5.6 [GSW]). *Let X be a star flow. Any invariant ergodic measure μ of the flow ϕ_t is a hyperbolic measure. Moreover, if μ is not the atomic measure on any singularity, then $supp(\mu) \cap H(P) \neq \emptyset$, where P is a periodic orbit with the index of μ, i.e., the number of negative Lyapunov exponents of μ(with multiplicity).*

of lemma 72. We argue by contradiction, assuming that the bundle E is not uniformly contracting for $h_E \cdot \psi^t_{\mathcal{N}}$ over $B(X, U)$ for every filtrating neighborhood U of the class C.

One deduces the following claim:

Claim. *Let $\tilde{C}(\sigma) \subset \tilde{\Lambda}(X)$ be the closure in $\mathbb{P}M$ of the lift of $C(\sigma) \setminus Sing(X)$. Then, for every $T > 0$, there exists an ergodic invariant measure μ_T whose support is contained in $\mathbb{P}^c_\sigma \cup \tilde{C}(\sigma)$ such that*

$$\int \log \| h^T_E . \psi^T_{\mathcal{N}} \mid_E \ \| \, d\mu(x) \geq 0 \, .$$

Proof. For all U, there exist an ergodic measure μ_T whose support is contained in $B(X, U)$ such that

$$\int \log \| h^T_E . \psi^T_{\mathcal{N}} \mid_E \ \| \, d\mu_T(x) \geq 0 \, .$$

But note that the class C, needs not to be a priori a maximal invariant set in a neighborhood U. We fix this by observing the fact that

$$\mathbb{P}^c_\sigma \cup \tilde{C}(\sigma) \subset B(X, U)$$

for any U as small as we want and actually we can choose a sequence of neighborhoods $\{ U_n \}_{n \in \mathbb{N}}$ such that $U_n \to C$ and therefore

$$\mathbb{P}^c_\sigma \cup \tilde{C}(\sigma) = \bigcap_{n \in \mathbb{N}} B(X, U_n) \, .$$

This defines a sequence of measures $\mu^n_T \to \mu^0_T$ such that

$$\int \log \| h^T_E . \psi^T_{\mathcal{N}} \mid_E \ \| \, d\mu^n_T(x) \geq 0 \, ,$$

and with supports converging to $\mathbb{P}^c_\sigma \cup \tilde{C}(\sigma)$. The resulting limit measure μ^0_T, whose support is contained in $\mathbb{P}^c_\sigma \cup \tilde{C}(\sigma)$, might not be hyperbolic but it is invariant. We can decompose it in sum of ergodic measures, and so if

$$\int \log \| h^T_E . \psi^T_{\mathcal{N}} \mid_E \ \| \, d\mu^0_T(x) \geq 0 \, ,$$

There must exist an ergodic measure μ_T, in the ergodic decomposition of μ_T^0,

$$\int \log \| h_E^T . \psi_{\mathcal{N}}^T \mid_E \ \| d\mu_T(x) \geq 0 \,,$$

and the support of μ_T is contained in $\mathbb{P}_\sigma^c \cup \tilde{C}(\sigma)$. CQFD

Recall that for generic star flows, every chain recurrence class in $B(X, U)$ is Hausdorff limit of periodic orbits of the same index and that satisfy the conclusion of Lemma 64. Let $\eta > 0$ and $T_0 > 0$ be given by Lemma 64. We consider an ergodic measure $\mu = \mu_T$ for some $T > T_0$.

Claim. *Let ν_n be a measure supported on a periodic orbits γ_n with period $\pi\gamma_n$ bigger than T , then $\int \log h_E^T d\nu_n(x) = 0$.*

Proof. By definition of h_E^T

$$\log h_E^T d\nu_n(x) = \log \Pi_{\sigma_i \in S_E} \| h_{\sigma_i}^T \| d\nu_n(x) \,,$$

so it suffices to prove the claim for a given $h_{\sigma_i}^T$. For every x in γ by the cocycle condition in lemma 27 we have that

$$\Pi_{i=0}^{(m\pi(\gamma)/T)-1} \qquad h_{\sigma_i}^T(\phi_{iT}^Y(x)) = h_{\sigma_i}^{(m\pi(\gamma)/T)-1}(x)$$

The norm of the vector field restricted to γ is bounded, and therefore $h_{\sigma_i}^{(m\pi(\gamma)/T)-1}(x)$ is bounded for $m \in \mathbb{N}$ going to infinity. Then this is also true for h_E^T. Since ν_n is an ergodic measure, we have that

$$\begin{aligned} \int \log h_E^T d\nu_n(x) &= \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{(m\pi(\gamma)/T)-1} \log\left(h_E^T(\phi_{iT}^Y(x))\right) \\ &= \lim_{m\to\infty} \frac{1}{m} \log\left(\Pi_{i=0}^{(m\pi(\gamma)/T)-1} h_E^T(\phi_{iT}^Y(x))\right) \\ &= \lim_{m\to\infty} \frac{1}{m} \log\left(h_E^{(m\pi(\gamma)/T)-1}(x)\right) \\ &= 0 \end{aligned}$$

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Claim. *There is a singular point σ_i so that μ is supported on $\mathbb{P}_{\sigma_i}^c$.*

Proof. Suppose that μ weights 0 on $\bigcup_{\sigma_i \in Sing(X)} \mathbb{P}^c_{\sigma_i}$. then μ projects on M on an ergodic measure ν supported on the class $C(\sigma)$ and such that ut weights 0 in the singularities, for which

$$\int \log \| h_E.\psi^T_{\mathcal{N}} \mid_E \ \| d\nu(x) \geq 0.$$

Recall that ψ^T is the linear Poincaré flow, and h^T_E can be defined as a function of $x \in M$ instead of as a function of $L \in \mathbb{P}M$ outside of an arbitrarily small neighborhood of the singularities.

However, as X is generic, the ergodic closing lemma implies that ν is the weak$*$-limit of measures ν_n supported on periodic orbits γ_n which converge for the Hausdorff distance to the support of ν. Therefore, for n large enough, the γ_n are contained in a small filtrating neighborhood of $C(\sigma)$ therefore satisfy

$$\int \log \| h^T_E.\psi^T \mid_E \ \| d\nu_n(x) \leq -\eta.$$

The map $\log \| h^T_E.\psi^T \mid_E \ \|$ is not continuous. Nevertheless, it is uniformly bounded and the unique discontinuity points are the singularities of X. These singularities have (by assumption) weight 0 for ν and thus admit neighborhoods with arbitrarily small weight. Out of such a neighborhood the map is continuous. One deduces that

$$\int \log \| h^T_E.\psi^T \mid_E \ \| d\nu(x) = \lim \int \log \| h_E.\psi^T \mid_E \ \| d\nu_n(x)$$

and therefore is strictly negative, contradicting the assumption. This contradiction proves the claim.

CQFD

On the other hand, Corollary 67 asserts that $h_E \cdot \psi_{\mathcal{N}}$ contracts uniformly the bundle E

— over the projective space $\mathbb{P}G^{cs}_{\sigma_i}$, for σ_i with $sv(\sigma_i) < 0$: note that, in this case, $\sigma_i \in S_E$ so that h_E coincides with h_{σ_i} on $\mathbb{P}G^{cs}_{\sigma_i}$;

— over $\mathbb{P}G^{cu}_{\sigma_i}$ for σ_i with $sv(\sigma_i) > 0$: note that, in this case $\sigma_i \notin S_E$ so that h^t_E is constant equal to 1 on $\mathbb{P}G^{cu}_{\sigma_i} \times \mathbb{R}$.

Recall that $\mathbb{P}^c_{\sigma_i}$ is contained in $\mathbb{P}G^{cs}_{\sigma_i}$ (resp. $\mathbb{P}G^{cu}_{\sigma_i}$) if $sv(\sigma_i) < 0$ (resp. $sv(\sigma_i) > 0$). One deduces that there is $T_1 > 0$ and $\varepsilon > 0$ so that

$$\log \| h_E.\psi^T_{\mathcal{N}} \mid_{E_L} \ \| \leq -\varepsilon, \quad \forall L \in \mathbb{P}^c_{\sigma_i} \text{ and } T > T_1.$$

Therefore the measures μ_T, for $T > \sup T_0, T_1$ cannot be supported on $\mathbb{P}^c_{\sigma_i}$, leading to a contradiction.

The expansion for F is proved analogously.

And this finishes the proof of Lemma 72 and therefore the proof of Lemma 63 and Theorem 6. CQFD

Chapter 8

Robustly Chain-transitive sets and Singular volume partial hyperbolicity

This chapter is dedicated to generalize the results in [BDP], where the authors prove that the robust chain transitivity of a set, comes along with a weak hyperbolic structure.

The main problem we need to deal with, in order to extend this result to singular flows, is the distortion of the contraction and expansion rates that occurs when the periodic orbits approach the singularities. For this we will use again the reparametrized linear Poincaré flow.

With this tool, our main work will be to find out which singularities need to be reparametrized, and then follow the strategy in [BDP]. That is, we show that the critical points in an open set of vector fields have the desired structure . Then we use the ergodic closing lemma, to argue that if the robust chain transitive set did not have the desired structure, then there must be a critical element in a perturbed vector field that does not have that structure either. This contradiction, gives us our result.

8.1 The set of periodic orbits

In a series of papers such as [BB], [BGV] and [BDP] [DPU] it is shown that if a set of periodic orbits, that has periodic orbits of arbitrarily long periods, does not have a dominated splitting, then there is a perturbation of the flow having a an infinite number of sinks and sources. We state two of these results below:

Lemma 74 ([BGV] Theorem 2.2). *Let $\mathcal{A}$ be a bounded linear cocycle over $\pi : E \to \sum$ where $\sum$ is a set of periodic orbits, containing periodic orbits of arbitrarily long periods. Then if $\mathcal{A}$ does not admit* *any dominated splitting, then there exist a perturbation $\mathcal{B}$ of $\mathcal{A}$ such that E is contracted or expanded by $\mathcal{B}$ along the orbit.*

Lemma 75 ([BDP] lemma 6.1)**.** *Let $\mathcal{A}$ be a bounded linear cocycle over $\pi : E \to \sum$ where $\sum$ is a set of periodic orbits, containing periodic orbits of arbitrarily long periods. Let $T(x)$ denote the period of $x \in \sum$ Suppose that $\mathcal{A}$*

— *$\mathcal{A}$ admits a dominated splitting $E = F_1 \oplus_{\prec} F_2$,*

— *$\mathcal{A}$ does not admit a dominated splitting of F_1.*

— *There exist a point p such that $\det(A^{T(p)} |_{F_1} (p)) > 1$*

then there exist a perturbation $\mathcal{B}$ of $\mathcal{A}$ and q in $\sum$ such that all Lyapunov exponents of $A^{T(q)} |_{F_1} (q)$ are positive.

Remark 33. Both of these theorems hold if we consider the linear Poincaré flow as the cocycle, and the time t of the flow as the diffeomorphism. Therefore the set of periodic orbits in a robustly chain transitive set Λ has dominated splitting and the finest possible of this dominated splittings is such that the extremal bundles contract and expand volume.

We choose now the set over which we will look at the extended linear Poincaré flow. This time our hypothesis give us information about an open set of vector fields, since we are talking about a robustly chain transitive set. For this situation we have:

$$\widetilde{\Lambda} = \overline{\{ < Y(x) > \in \mathbb{P}M \text{ such that } x \in U \cap Per(Y) \text{ and } Y \in \mathcal{U} \}}.$$

We will prove that For an open an dense set of $\mathcal{X}^1 M$, any X such that Λ is robustly chain tranistive is volume singular partial hyperbolic over $\widetilde{\Lambda}$. From theorem 3 we have that there is an open and dense set of vector fields in which this is implies that X is volume singular partial hyperbolic over $B(X, U)$ and in fact, this two open and dense sets of $\mathcal{X}^1 M$ are the same.

Lemma 76. *There is an open and dense subset $\mathcal{G}\mathcal{X}^1 M$ such that for any $Y \in \mathcal{G}$ a vector field having a robustly chain transitive set Λ in $\mathcal{U} \subset \mathcal{X}^1 M$. Then the set $\widetilde{\Lambda} \subset \mathbb{P}M$ admits a finest dominated splitting of the normal bundle, for the extended linear Poincaré flow.*

Proof. From theorem 18 we can find a dense set $\mathcal{G} \subset \mathcal{X}^1 M$ in which any Y is such that the set Λ is the hausdorff limit of the periodic orbits of the Y. Since Λ is robustly chain transitive this periodic orbits must be related, so for Y_n, Λ is the closure of $U \cap Per(Y)$. From lemma 74 the set $\{ < Y(x) > \in \mathbb{P}M$ such that $x \in \Lambda \}$ has a uniform finest dominated splitting for the extended linear Poincaré flow (note that the projection to M here is one to one). Therefore, the extended linear Poincaré flow has dominated splitting over the set

$$\overline{\left\{ < Y_(x) >\in \mathbb{P}M \text{ such that } x \in \Lambda \right\}},$$

since a uniform dominated splitting extends to the closure.

Since the dominated splitting is a robust property then the set

$$\widetilde{\Lambda} = \overline{\{ < Y(x) >\in \mathbb{P}M \text{ such that } x \in \Lambda \ \text{ and } \ Y \in \mathcal{U} \}},$$

has a dominated splitting.

CQFD

8.2 Analyzing the singulatiries

Recall that for a hyperbolic singularity we note the center space as:

$$E^c_\sigma = E_{s1} \oplus \cdots \oplus E_{sk} \oplus E_{u1} \oplus \cdots \oplus E_{ul}$$

Lemma 77. *Let σ be a singularity of Λ, a robustly transitive chain recurrence class. Let $\Gamma = Orb(x)$ be a homoclinic orbit associated to σ . Assume as well that:*

— *there exists a sequence of vector fields X_n converging to X in the C^1 topology*

— *there exist a sequence of periodic orbit γ_n of X_n such that γ_n converges to Γ in the Hausdorff topology.*

— *there is a sequence of points $q_n \in \gamma_n$ such that $X_n(q_n) \to u$ is in $E^{u1} \oplus \cdots \oplus E^{ul}$.*

— *we call $L_u =< u >$ and we have that $dim(\mathcal{N}^u_{L_u}) = h$*

— *There is a sequence of points $p_n \in \gamma_n$ such that $< X_n(p_n) >\to v$ is in $E_{s1} \oplus \cdots \oplus E_{sk}$*

— *we call $L_v =< v >$ and we have that $dim(\mathcal{N}^s_{L_v}) = n$.*

Then there is a space $E \subset T_\sigma M$ such that E contracts volume and has dimension $n+1$, and a $F \subset T_\sigma M$ such that F expands volume and has dimension $h+1$

Proof. Let us recall that from Remark 33 the splitting

$$\mathcal{N}_L = \mathcal{N}^s \oplus \mathcal{N}^1 \oplus \cdots \oplus \mathcal{N}^k \oplus \mathcal{N}^u$$

where L is a direction over the set of periodic orbits of Λ is Volume partial hyperbolic. This means that $\mathcal{N}^s$ contracts volume and $\mathcal{N}^u$ expand volume uniformly in the period.

Since in the period $X_n(xn)$ does not contract or expand (for any x_n) in gamma, then $\mathcal{N}^s_L \oplus < X_n(xn) >$ contracts volume and $< X_n(x_n) > \oplus \mathcal{N}^u_L$ expands volume. Since γ_n tends to the homoclinic loop associated to Γ, their periods must tend to infinity with n. For n large enough, and from the contraction of volume we have there exist some constants ν and T

$$\prod_{i=0}^{\llcorner T_{\gamma_n}/T \lrcorner -1} \det(D\phi^{iT}(x) \mid_{\mathcal{N}_L^s \oplus <X_n(_n)>}) \leq e^{-\nu T_{\gamma_n}} ,$$

where T_{γ_n} is the period of γ_n. Then for any γ_n,takeing $T = 1$, Pliss Lemma **??** gives some point $p_n \in \gamma_n$ satisfying

$$\frac{1}{k}\sum_{i=0}^{k/1} \log(\det(D\phi^1 \mid_{D\phi^i(\mathcal{N}_L^s \oplus <X_n(p_n)>)})) \leq -\nu .$$

Since the orbits Γ and γ_n for n large enough spend only uniformly bounded time outside of any neighborhood of the singularity that we call U_σ. Since the periods of this orbits tend to infinity with n and the pliss points are distributed proportionably in the time, there must be a sequence of Pliss points Assume p_n tends to σ. One can assume $\mathcal{N}_L^s \oplus < X_n(xn) > \to E(\sigma)$, and since σ is accumulated by Pliss points, again we have that:

$$\frac{1}{k}\sum_{i=0}^{k/1} \log(\det(D\phi^1 \mid_{D\phi^i(E(\sigma))})) \leq -\nu ,$$

This shows that $E(\sigma)$ contracts volume, and the proof is analogous for F CQFD

The following corollary is a consequence of Lemma77.

Corollary 78. *Let σ be a singularity of Λ, a robustly chain transitive class, with a finest dominated splitting over $\widetilde{\Lambda}$*

$$\mathcal{N}_L = \mathcal{N}^s \oplus \mathcal{N}^1 \oplus \cdots \oplus \mathcal{N}^k \oplus \mathcal{N}^u .$$

Suppose that $n = dim(\mathcal{N}^s) > dim(E^{ss})$ then there is a $n+1$ dimensional space E that contracts volume. Moreover $E^{ss} \oplus E^c \subset E$

Proof. we can find a vector field Y, that is $\epsilon - C^1$ close to X and such that the singularities of Y have the same Lyapunov exponents as the ones in X but Y is Kupka-Smale. Then we con perturb again so that that there is $\Gamma = Orb(x)$ a homoclinic orbit associated to σ, by the connecting lemma17. Now theorem18 allow us to find a sequence of vectorfields

- — there exists a sequence of star vector fields Y_n converging to Y in the C^1 topology
- — there exist a sequence of periodic orbit γ_n of Y_n such that γ_n converges to Γ in the Hausdorff topology.
- — there is a sequence of points $q_n \in \gamma_n$ such that $Y_n(q_n) \to u$ is in $E^{u1} \oplus \cdots \oplus E^{ul}$.

— we call $L_u =< u >$ and we have that $dim(\mathcal{N}^u_{L_u}) = h$
— There is a sequence of points $p_n \in \gamma_n$ such that $< Y_n(p_n) > \to v$ is in $E^{s1} \oplus \cdots \oplus E^{sn}$
— we call $L_v =< v >$ and we have that $dim(\mathcal{N}^s_{L_v}) = n$.

Since Y is now in the conditions of Lemma 77 then there is an invariant space E of dimension $n+1$ that contracts volume. Since from Corollary 33 $dim(E^{ss} \oplus E^c) \leq n+1$, then $E^{ss} \oplus E^c \subset E$.

CQFD

Corollary 79. *Let σ be a singularity of Λ, a robustly chain transitive class, with a finest dominated splitting over $\widetilde{\Lambda}$*

$$\mathcal{N}_L = \mathcal{N}^s \oplus \mathcal{N}^1 \oplus \cdots \oplus \mathcal{N}^k \oplus \mathcal{N}^u .$$

Suppose that $n = dim(\mathcal{N}^s) > dim(E^{ss})$ then $E^{cs} = E^{ss} \oplus E^c_\sigma$ contracts volume.

8.2.1 The reparametrizing cocycle

Now we can choose the set of singularities over which we will reparametrize.

Definition 34. Let X be a C^1 vector field, such that there is an open set U such the maximal invariant set in U is a robustly chain transitive chain recurrence class. Suppose as well that the singularities of X are all hyperbolic and that the dimension of the center space of its singularities are locally minimal. We ask as well that there is a finest finest dominated splitting for the pre extended maximal invariant set $\widetilde{\Lambda}$,

$$\mathcal{N}_L = \mathcal{N}^s_L \oplus \cdots \oplus \mathcal{N}^U_L .$$

We define S_{Ec} the set of singularities

$$S_{Ec} = \{ \sigma \in Sing(X) \cap U \text{such that } dim(\mathcal{N}^s_L) > E^{ss}_\sigma \} .$$

Similarly we define the set S_{Fc} the set of singularities

$$S_{Fc} = \{ \sigma \in Sing(X) \cap U \text{such that } dim(\mathcal{N}^u_L) > E^{uu}_\sigma \} .$$

Note that this sets are disjoint

Definition 35. We define :
— The center-stable reparametrizing cocycle as:

$$h_{Ec} = \Pi_{\sigma \in S_{Ec}} h_\sigma .$$

— The center-stable reparametrizing cocycle as:

$$h_{Fc} = \Pi_{\sigma\in S_{Fc}} h_\sigma.$$

8.3 Proof of the main theorem

We aim now to prove theorem 9. The proof is very similar to the proof in [Ma2]. In fact is an adaptation to flows of the proof of theorem 4 in [BDP]. We already used this strategy in 72. The idea is to argue by contradiction and show that if there is no uniform volume expansion in the extremal bundle, then there is a closed orbit orbit of a sufficiently close vector field that contracts volume in the extremal bundle. This could be a periodic orbit or a singularity, but sections 3.2 and 8.1 show us that this is not possible. The following proposition is equivalent to lemma 6.5 form [BDP], and the proof is analogous.

Lemma 80. *Let $X \in \mathcal{X}^1 M$ be a vector field, L_a a maximal invariant in a filtrating neighborhood $U \subset M$ and set $\widetilde{\Lambda}$. Suppose there is a dominated splitting $E \oplus_{\prec} F$ over $\widetilde{\Lambda}$ for the reparametrized linear Poincaré flow, Ψ. Then if the Jacovian of Ψ restricted to E is not bounded from above by one, then for every T there is a Ψ_T invariant measure ν such that*

$$\int log|\, J(\Psi_T, E)\,|\; d\nu\; \geq 0\,.$$

Now we want to show that if Ψ does not contract volume on the most dominated bundle of the finest dominated splitting in $B(X,U)$, then, the measure ν from the previous lemma is not supported on the directions that are over the singularities.

The following lemma is a consequence of corollary 78.

Lemma 81. *Let $X \in \mathcal{X}^1 M$ be a vector field, L_a a maximal invariant in a filtrating neighborhood $U \subset M$, with a singularity σ and the set $\widetilde{\Lambda}$. Suppose there is a finest dominated splitting*

$$\mathcal{N}_L = \mathcal{N}_L^s \oplus_{\prec} \cdots \oplus_{\prec} \mathcal{N}_L^u$$

over $\widetilde{\Lambda}$ for the reparametrized linear Poincaré flow, Ψ. Any Ψ_T invariant measure ν supported in $\mathbb{P}^c_\sigma$ is such that

$$\int log|\, J(\Psi_T, \mathcal{N}_L^s)\,|\; d\nu\; < 0\,.$$

Proof. Let us start by supposing that $\sigma \in S_E$ and $dim(\mathcal{N}^s) \leq E^{ss}$. In this case, we can include $\mathcal{N}^s \subset T\sigma M$ as a subspace of E^{ss}. Then $\mathcal{N}^s$ contracts uniformly for the tangent space and for the extended linear Poincaré flow. Note that in this case the reparametrized linear

Poincaré flow and the extended linear Poincaré flow are equal in restriction to $dim(\mathcal{N}^s)$ at the directions over σ. Suppose that $\sigma \in S_{Ec}$ and $dim(\mathcal{N}^s) \geq E^{ss}$, then corollary 78 implies that $E^c s$ contracts volume. The reparametrized linear Poincaré flow at the directions over σ is $h^T_{Ec}.\psi^T(L)$ where $h^T_{Ec} = \frac{\| d\phi^t(u) \|}{\| u \|}$ for a non vanishing vector u in the direction of L. Then

$$| J(\psi_T, \mathcal{N}^s_L) | \frac{\| d\phi^t(u) \|}{\| u \|} = | J(d\phi_T, E^{cs}) | \,.$$

In any case the same construction as in Lemma 80 allow us to conclude. CQFD

The flowing lemma is the only missing piece for Theorem 9. Till now we have from Lemma 80 that if there is no volume contraction of $\mathcal{N}^S_L$, then there is a measure showing this lack of contraction. From Lemma 81 we also know that this measure can not be supported over a singularity. Finally the next lemma uses the ergodic closing lemma to prove that if a measure was showing the lack of contraction, then it would be supported on a singularity contradicting the previous lemma. So by contradiction the following lemma implies Theorem 9.

Lemma 82. *Let $X \in \mathcal{X}^1 M$ be a vector field, L_a a maximal invariant in a filtrating neighborhood $U \subset M$ and the set $\widetilde{\Lambda}$. Suppose there is a finest dominated splitting*

$$\mathcal{N}_L = \mathcal{N}^s_L \oplus_{\prec} \cdots \oplus_{\prec} \mathcal{N}^u_L$$

over $B(X, U)$ for the reparametrized linear Poincaré flow, Ψ. If there is a Ψ_T invariant measure ν not supported in $\mathbb{P}^c_\sigma$ then if

$$\int log | J(\Psi_T, \mathcal{N}^s_L) | \, d\nu \geq 0\,,$$

then the measure must be supported on a singularity

Proof.

Claim. *Let ν_n be a measure supported on a periodic orbits γ_n with period $\pi\gamma_n$ bigger than T , then $\int \log h^T_{Ec} d\nu_n(x) = 0$.*

Proof. By definition of h^T_{Ec}

$$\log h^T_{Ec}\, d\nu_n(x) = \log \Pi_{\sigma_i \in S_{Ec}} \| h^T_{\sigma_i} \| \, d\nu_n(x)\,,$$

so it suffices to prove the claim for a given $h^T_{\sigma_i}$. For every x in γ by the cocycle condition in Lemma 27 we have that:

$$\Pi_{i=0}^{(m\pi(\gamma)/T)-1} \qquad h_{\sigma_i}^T(\phi_{iT}^Y(x)) = h_{\sigma_i}^{(m\pi(\gamma)/T)-1}(x)$$

The norm of the vector field restricted to γ is bounded, and therefore $h_{\sigma_i}^{(m\pi(\gamma)/T)-1}(x)$ is bounded for $m \in \mathbb{N}$ going to infinity. Then this is also true for h_{Ec}^T. Since ν_n is an ergodic measure, we have that

$$\begin{aligned}
\int \log h_{Ec}^T d\nu_n(x) &= \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{(m\pi(\gamma)/T)-1} \log\left(h_{Ec}^T(\phi_{iT}^Y(x))\right) \\
&= \lim_{m\to\infty} \frac{1}{m} \log\left(\Pi_{i=0}^{(m\pi(\gamma)/T)-1} h_{Ec}^T(\phi_{iT}^Y(x))\right) \\
&= \lim_{m\to\infty} \frac{1}{m} \log\left(h_{Ec}^{(m\pi(\gamma)/T)-1}(x)\right) \\
&= 0
\end{aligned}$$

CQFD

Suppose that μ weights 0 on $\bigcup_{\sigma_i\in Sing(X)} \mathbb{P}^c_{\sigma_i}$. then μ projects on M on an ergodic measure ν supported on the class $C(\sigma)$ and such that ut weights 0 in the singularities, for which

$$\int \log |\, J(h_E.\psi_{\mathcal{N}}^T, \mathcal{N}^s)\,|\, d\nu(x) \geq 0.$$

Recall that ψ^T is the linear Poincaré flow, and h_{Ec}^T can be defined as a function of $x \in M$ instead of as a function of $L \in \mathbb{P}M$ outside of an arbitrarily small neighborhood of the singularities.

However, as X is generic, the ergodic closing lemma implies that ν is the weak$*$-limit of measures ν_n supported on periodic orbits γ_n which converge for the Hausdorff distance to the support of ν. Therefore, for n large enough, the γ_n are contained in Λ and from remark 33 we know that

$$\int \log |\, J(h_{Ec}.\psi_{\mathcal{N}}^T, \mathcal{N}^s)\,|\, d\nu_n(x) \leq -\eta.$$

The map $\log |\, J(h_{Ec}.\psi_{\mathcal{N}}^T, \mathcal{N}^s)\,|$ is not continuous. Nevertheless, it is uniformly bounded and the unique discontinuity points are the singularities of X. These singularities have (by assumption) weight 0 for ν and thus admit neighborhoods with arbitrarily small weight. Out of such a neighborhood the map is continuous. One deduces that

$$\int \log |\, J(h_{Ec}.\psi_{\mathcal{N}}^T, \mathcal{N}^s)\,|\, d\nu(x) = \lim \int \log |\, J(h_{Ec}.\psi_{\mathcal{N}}^T, \mathcal{N}^s)\,|\, d\nu_n(x)$$

and therefore is strictly negative, contradicting the assumption. CQFD

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