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An Introductions to Dynamic Optimization

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Contents

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Acknowledgments

These notes are the result of courses dictated in the Department of Economy of the University of Siena, in the Faculty of Sciences of the Pontifical Catholic University of Peru and specially in the Seminary of Mathematical Economy of the Department of Economy of the Social Sciences School of Uruguay.

They are dedicated mainly to economists with taste by the rigorous formalization of the economic theory, and to mathematicians interested in the applications from their knowledge to the challenging field of the modern economic theory. The basic reference for the theory of control itself, is the excellent and classic text of [Lee, E.; Markus, L.]. The applications to the economic theory can be found in diverse texts and articles, among others in the mentioned ones in the bibliography.

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The errors of all type and omissions are exclusive responsibility of the author.

A mis Marías, France y Vari, con todo mi cariño:
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1 Introduction

The goal of these notes is to provide a description of the optimal theory of control and its applications to the economy. These applications are analyzed mathematically. Mathematics provide compact ways to express ideas and are long-range tools to analyze them. In order to obtain these objectives a deep knowledge of the tool is necessary, therefore we will offer the proofs of the theorems and the main subjects related to the applications.

- Consider the problem of shooting a guided missile to intercept an airplane. The problem is to obtain an optimal trajectory that minimizes the time for the missile to reach the airplane. Clearly this trajectory can be controlled by a number of variables. These variables are represented as a vector in R^n and hence the problem is to obtain a trajectory (the states of the system) by choosing a function (a control function) so as to maximize or minimize a certain objective.

- The economy of a typical capitalistic country is a system made up in part of the populations (as consumers and as producers) , companies, material goods, production facilities, cash, credit available, and so on. The *state* of the system can be thought of as a massive collection of data wages and salaries, profits, losses, sales of goods and services, investments, unemployment, welfare costs, the inflation rate, gold and currency holdings, and foreign trade. The central government can influence the state of this system by using several *controls,* notably the prime rate, taxation policy, and persuasion regarding wage and price settlements.

2 Some Examples of optimal control problems

Only continuous deterministic process are investigated in these notes. We shall here introduce the concepts and methods of optimal control theory, we begin considering several particular examples:

Example 1 Control of a mechanism along a smooth track.

Consider a mechanism as a cart or a trolley, of mass m *which move along a track with negligible friction. The position coordinate at time* t *is determined by Newtons law:*

$$m\ddot{x} = u(t),$$

where $u(t)$ *is the external controlling force. Suppose that initial position and velocity are* (x_0, y_0)*. We consider the problem of stopping the trolley at a prescribed target:* $x = y = 0$ *in minimal time possible by means of (possibly discontinuous force)* $u(t)$ *subject to the restraint* $|u(t)| \leq 1$.

For convenience choose $m = 1$ and we write the Newton equation:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= u(t) \end{aligned} \qquad (\mathcal{S})$$

or in matrix notation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

or $\dot{x} = Ax + bu$.

Fix time t_1 and consider all the various possible controllers $u(t)$ on $0 \leq t \leq t_1$ with $|u(t)| \leq 1$. By direct substitution we can see that the solution is:

$$x(t) = x_0 + y_0 t + \int_0^t \left[\int_0^s u(\sigma) d\sigma\right] ds$$

$$y(t) = y_0 + \int_0^t u(\sigma) d\sigma$$

or: $x(t) = e^{At}x_0 + e^{At} \int_0^t e^{-As} bu(s) ds$.

Define the set $K(t_1)$ in the phase plane, to be the totality of all end points of all responses which initiate a x_0, we will prove later that $K(t_1)$ is a closed, bounded and a convex set, that varies continuously with the end time t_1.

The minimal time $t = t^*$ is determined as the first time at which $K(t)$ contains the target $(0, 0)$. It can be proved that $(0, 0)$ lies on the boundary of the set $K(t^*)$. See subsection (9).

The optimal response $\bar{x}^*(t) = (x^*(t), y^*(t))$ leads to the origin at $t = t^*$. And the optimal controller $u^*(t)$ is the controller that produce the optimal response.

Let $\eta(t^*) = (\eta_1(t^*), \eta_2(t^*))$ be a unit vector at the origin that is an outward normal for the convex set $k(t^*)$. Then for each response $x(t)$ we must have:

$$\eta(t^*)\bar{x}(t^*) \leq 0,$$

that is, the vector $x(t^*)$, has no positive component along the direction of $\eta(t^*)$. , observe that $x^*(t^*) = \mathbf{0}$, this means that:

$$\eta_1(t^*)x_1^*(t^*) + \eta_2(t^*)y^*(t^*) = \max_{(x,y)\in K(t^*)} \eta_1(t^*)x + \eta_2(t^*)y.$$

This inequality is called the **maximal principle**

Using the explicit integral expressions we consider de maximal principle:

$$\eta_1(t^*) \left[x_0 + y_0 t^* + \int_0^{t^*} \int_0^s u(\sigma) d\sigma ds\right] + \eta_2(t^*) \left[y_0 + \int_0^{t^*} u(\sigma) d\sigma\right].$$

When we disregard all terms not involving $u(t)$ the expression that must be maximized is:

$$\eta_1(t^*)\int_0^{t^*}\int_0^s u(\sigma)d\sigma ds + \eta_2(t^*)\left[\int_0^{t^*} u(\sigma)d\sigma\right].$$

From the equality:

$$\int_0^t\int_0^s u(\sigma)d\sigma ds = \int_0^t (t-\sigma)u(\sigma)d\sigma$$

and doing;

$$\eta_1(s) = \eta_1(t^*), \eta_2(s) = \eta_1(t^*)(t^* - s) + \eta_2(t^*)$$

on the interval: $0 \leq s \leq t^*$, we must maximize:

$$\int_0^{t^*} \eta_2(s)u(s)ds.$$

It is clear that the maximun for the integral, recalling that $|u(t)| \leq 1$, is achieved only by the controller:

$$u^*(t) = sign\ \eta_2(t)\ \ on\ \ 0 \leq t \leq t^*.$$

Therefore the optimal controller $u^*(t)$ assume only the values $+1$ and -1, except where it switches between these, precisely at the zeroes of the unknown function $\eta_2(t)$. Note that from the definition of $\bar{\eta}(t)$ it follows that $\ddot{\eta}_2(t) = 0$. Then $\eta_2(t)$ is a linear polynomial in t. So, $u^*(t)$ can have at most one zero.

The optimal response from x_0 to the origin must follow an arc of a solution of the extremal differential system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -1 \end{aligned} \qquad (\mathcal{S}-)$$

and then an arc of a solution of the extremal:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 1 \end{aligned} \qquad (\mathcal{S}+)$$

Let us construct all possible extremal responses. To do this we follows the method of backing out of the target. Choose a unit vector $\eta(0) = (\eta_1(0), \eta_2(0))$ and use this as initial date to determine the solution (backward on time) of:

$$\dot{x} = y$$

$$\dot{y} = sgn\ \eta_2(t)$$

If we consider $\eta_2(t) > 0$ on $t < 0$ we obtain the curve trough the origin

$$\Gamma_+ : 2x = y^2, \quad for \;\; y(= \dot{x}) \leq 0.$$

Similarly if we take: $\eta_2(0) < 0$ we trace the response along

$$\Gamma_- : -2x = y^2, \quad for \;\; y(= \dot{x}) \geq 0.$$

For arbitrary $\eta(0)$ we follow Γ_+ or Γ_- respectively if $\eta_2(0)$ is positive or negative, until $\bar{t}$: $\eta(\bar{t}) = 0$ then we follow the appropriate solution of $\mathcal{S}-$ or respectively $\mathcal{S}+$.

The curve consisting of Γ_- and Γ_+ is called the switching locus W. In this example

$$y = W(x) = \begin{cases} -\sqrt{2x} & for \;\; x \geq 0 \\ +\sqrt{-2x} & for \;\; x < 0 \end{cases}$$

We define the synthesizer by;

$$\Phi(x,y) = \begin{cases} -1 \; if \; y > W(x) \quad or \;\; if \quad (x,y) \neq (0,0) \;\; lies \; on \; \Gamma_- \\ \\ 0 \;\; if \; x = y = 0 \\ \\ +1 \; if \; y < W(x) \quad or \;\; if \quad (x,y) \neq (0,0) \;\; lies \; on \; \Gamma_+ \end{cases}$$

Then the optimal response from any initial state (x_0, y_0) to the origin is just the solution of $\ddot{x} = \Phi(x,y)$, and the optimal controller $u^*(t) = \Phi(x, \dot{x})$.

Example 2 *Consider an economy that at any time has some amount of capital $K(t)$, and labor $L(t)$, these are combined to produce output $Y(t)$. Suppose that labor grows exponentially $L(t) = L_0 e^{nt}$, where $L(0) = L_0$ denotes the initial amount of labor. The production function takes the form: $Y(t) = F(K(t), L(t))$, where t denotes time. Suppose that the production function has constant returns to scale in its two arguments. That is: $F(cK, cL) = cF(K,L), \quad c > 0$. Define $k = K/L$ the capital per unit of labor, the assumption of constant returns to scale allow us to work with the production function in the form $f(k) = \frac{1}{L}F(K(t), L(t)) = F(\frac{K}{L}, 1)$.*

Suppose that output is divided between consumption and investment. The factor devoted to investment is s is exogenously determined. One unit of output devoted to investment yield one unit of new capital. In addition existing capital depreciates at rate δ, where $\delta < a$. As usual, capital stock increases with investments and decreases with depreciation. So, the dynamics of the capital is given by the equation:

$$\dot{K(t)} = sY(t) - \delta K(t).$$

Suppose that $F(K(t), L(t)) = aK(t)L(t)$ *So, we obtain*

$$\dot{k} = sy(t) + (\delta - n)k$$

where $y(t) = Y(t)/L(t)$. *It follows that*

$$\dot{k} = (sa - \delta + n)k. \tag{1}$$

Suppose that the central planner wishes to maintain the economy in an steady state: $k(t) = \bar{k}$ *and that any deviation of this value be corrected in minimal time. The political mechanism to obtain this target is the saving rate* $0 \leq s \leq 1$.

The solution of 1 is $k(t) = k_0 e^{(sa+n-\delta)t}$, here k_0 is the value of capital at $t = 0$. Consider the case where $n < \delta$. Then if:

1. $k_0 < \bar{k}$ the planner choose $s(t) = 1$ until $k(t) = \bar{k}$ this occurs at time $t^* = [log\frac{\bar{k}}{k_0}]\frac{1}{a+n-\delta}$ and then the planner fixes $s(t) = \frac{-n+\delta}{a}$ for all $t \geq t^*$. We obtain that the optimal saving rate is

$$s^*(t) = \begin{cases} 1 & t < t^* \\ \frac{-n+\delta}{a} & t \geq t^* \end{cases}$$

2. $k_0 > \bar{k}$. Analogously:

$$s^*(t) = \begin{cases} 0 & t < t^{**} \\ \frac{-n+\delta}{a} & t \geq t^{**} \end{cases}$$

   Where $t^{**} = [log\frac{\bar{k}}{k_0}]\frac{1}{n-\delta}$.

We define the synthesizer by:

$$\psi(k) = \begin{cases} 1 & k < \bar{k} \\ \\ \frac{-n+\delta}{a} & k = \bar{k} \\ 0 & k < \bar{k} \end{cases}$$

3 A mathematical formulation

We now give a precise mathematical formulation of the type of control problem we will be discussing.

3.1 Some of notation

Let m and n be natural numbers, and let R stand for the real numbers. If x, is a vector in R^n, we denote by $i-th$ its component x_i. We define x^{tr} to be the transpose of x, and introduce a dot product and two norms:

$$xy = x^{tr}y = \sum_{i=1}^{n} x^i y^i,$$

$$|x| = \sum_{i=1}^{n} |x^i|, \|x\| = (xx)^{\frac{1}{2}}.$$

Remark 3 *We say that a property is fulfilled almost surely (a.s) or almost everywhere (a.e.), if it fails in a set of null measurement*

A set $N \in R^n$ is called a null set , if N can be covered by a countable union of n-cubes whose total n-volume is less than an arbitrarily prescribed $\epsilon > 0$.

Given a set C we will denote its interior by $int[C]$ and the points belonging to its frontier will be denoted by $Fr[C]$.

The most general control process considered here is described by the following dates:

3.2 The optimal control problem: a survey

1. *The plant of the process.* It relates the state or response $x(t)$ to the input or control $u(t)$ by a ordinary differential system $\dot{x}_i(t) = f_i(t, x(t), u(t))$ $i = 1, ...n$. If the variable t does not appear explicitly the process is called autonomous, otherwise it is called non-autonomous. A measurable function u defined on an interval $[t_0, t_1]$ with range in R^m, is said to be a *control* if there exists an absolutely continuous function x defined on $[t_0, t_1]$ with range in R^n such that is a solution of the system of differential equations :

$$\dot{x} = f(t, x, u).$$

   Fixed the initial position $x(0) = x_0$, each controller $u(t)$, defines a unique response $x(t) \in R^n$.

2. *An initial point or state* x_0 and *a target set $\mathcal{T}$ are prescribed.* The initial point x_0 is a known vector in phase space. In a real physical process x_0 and the response $x(t)$ describe position, velocities or other dates that can be measured by appropriate instruments. In economics these vectors describe the initial capital $k(0)$ and its evolution $K(t)$, or initial consumption of some good and its intertemporal evolution. In other cases they can describe prices or other data that can be measured by the agents of the economy. The target set represents

the desirable states at the end of the process. Sometimes this is a moving set $\mathcal{T}(t) \in R^n$ at each t.

3. The class of admissible controllers Δ usually consists of measurable functions [1] $u(t)$ on various time intervals $t_0 \leq t \leq t_1$. Various additional restrictions are often imposed on the function comprising Δ. For example, the condition $u(t) \subset \Omega$, where Ω is a fixed compact and convex restraint set in R^n is usual. Also the initial or final time for the duration of the controllers is sometimes prescribed.

4. The cost functional or objective functional, is an accepted quantitative criterion for the efficiency of each admissible controller $u(t)$. Often the cost functional is described as: $C(u) = \int_{t_0}^{t_1} f^0(t, x(t), u(t))dt$. The optimal control problem consists in maximizing or minimizing a cost functional choosing for this an admissible control in such way that the response $x(t)$ verifies the initial state at $t = t_0$ and $x(t_1) \in G(t_1)$.

Definition 4 *A controller $u^*(t)$ in the admissible class Δ is called* **optimal** *in case $C(u^*) \leq C(u)$ for all $u(t) \in \Delta$.*

If $f^0(t, x(t), u(t)) \equiv 1$, then $C(u) = t_1 - t_0$ and we have the minimal time problem.

Definition 5 *Each choice of the control, $u(\cdot)$ generates a response $x(t) = x(t, x_0, u(\cdot))$, if the response reaches the target at some t_1 then $u(\cdot)$ is a* successful control. *We define:*

$$\mathcal{S}(t_1, x_0) = \{u(\cdot) \in \Omega : \textit{ there exists } t_1 \geq 0 : x(t_1, x_0, u(\cdot)) \in \mathcal{T}(t_1)\}$$

Remark 6 Summarizing: *We assume that the dynamic of the system, that is the evolution of the state $x(t)$ under a given control $u(t)$, is determined by a system of ordinary differential equation: $\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0$, where x_0 is the initial state, $f \in C^1$. Also restraint set Ω and a target $\mathcal{T}$ are given.*

In each optimal control problem our ultimate goal is to synthesize the optimal control by an appropriately designed feedback loop. A feedback control can often correct for unpredictable variations in the environment of the plant or irregularities in the process.

[1] A real function $h(t)$ on a real interval $\mathcal{I}$ is called measurable in case for all real α and β, the set $\{t \in \mathcal{I} \textit{ and } \alpha < h(t) < \beta\}$ is measurable in R^1. If $h(t)$ is measurable on $\mathcal{I}$ we can define the Lebesgue integral, $H(t) = \int_{\mathcal{I}} h(t)dt$ by considering appropriate limits of approximating sums. If $h(t)$ is piecewise continuous the Lebesgue integral is the same that the usual Riemann integral. The indefinite integral $H(t) = \int_{\mathcal{I}} h(t)dt$ defines an absolutely continuous function. Recall that the measurable sets of R^n are defined as the members of the smallest family of sets in R^n that contains all open and closed sets, the null sets of R^n and also every countable union and intersection of its members and its complements.

As we shall show in the next example, given a system of state equations together with end conditions and control constraints there is not guarantee that the set of admissible pairs, control and its corresponding response, $u(t), x(t)$, its not void.

Example 7 *Let x be one dimensional. Let the state equation be $\dot{x} = u(t)$. Let $x(0) = 0$, and $x(1) = 2$ be the initial and final point. Let $\Omega = \{u : |u| \leq 1\}$ be the set of admissible controllers.*

The trajectory is given by $x(t) = \int_0^t u(s)ds$. These pairs are no admissible, because we need $x(1) = 2$ but if $u \in \Omega$ then $|u| \leq 1$.

3.3 The controllable set

Definition 8 *We define the* controllable set, *at time t_1 as the set of initial states x_0 that can be steered, using a bounded measurable controller, to the target set in time t_1.*

$$\mathcal{C}(x_0) = \cup_{t_1>0}\mathcal{C}(t_1, x_0),$$

where

$$\mathcal{C}(t_1, x_0) = \{x_0 \in R^n : \text{ there exists } u(\cdot) \in \Omega_b : x(t_1, x_0, u(\cdot)) \in \mathcal{T}(t_1)\}$$

where: Ω_b is the class of bounded measurable controls on $[t_0, t_1]$. In the case in that $x_0 = 0$ we will use the notation $C(t)$ for the controllable set at time t from 0.

An autonomous control process $\dot{x} = f(x, u)$ is said *completely controllable* in the cases in which for each pair of points x_0 and x_1 in R^n there exists a bounded measurable controller $u(t)$ on some finite interval $0 \leq t \leq t_1$ such that the corresponding response $x(t)$ steers $x(0) = x_0$ to $x(t_1) = x_1$.

In some cases we will consider $\mathcal{T}(t) \equiv 0 \in R^n$. In these cases the controllable set is called the set of *null controllability.*

The control problem is to determine those x_0 and $u(\cdot) \in \Delta$ such that the associated response satisfies $x(t_1) \in \mathcal{T}(t_1)$ for some $t_1 > 0$, we then say that **the control** *u steers x_0 to the target.*

3.4 The set of attainability

Consider a control process $\dot{x} = f(x, u)$ with restraint set Ω, initial state x_0, Let $x(t)$ be the corresponding responses initiating at $x(t_0) = x_0$. The **attainable set** (or reachable) $\mathcal{K}(t_1, \mathbf{x_0})$ is the set of all end points $x(t_1)$ from the initial point x_0 in time t_1. When x_0 is given we will denote this set by $K(t_1)$.

3.4.1 Continuity of the set of attainability

Suppose that $K(t)$ is a continuously moving set on $\tau_0 \leq t \leq \tau_1$. That is, for each such instant t we designate a nonempty set $K(t)$ in R^n. The continuity of $K(t)$ as a function of the real variable t is defined in terms of the following concept of the distance from $K(t)$ to $K(t')$:

$$dist(K(t), K(t')) = max\left[\max_{P\in K(t)} dist(P, K(t')), \max_{P'\in K(t')} dist(P', K(t)).\right]$$

Thus, $dist(K(t), K(t'))$ is a continuous function at t if for each given $\epsilon > 0$ there exist $\delta > 0$ such that $dist(K(t), K(t')) < \epsilon$ whenever $|t - t'| < \delta$.

Remark 9 *Now we will prove that, if t^* is the first time at which the attainable set $K(t)$ contain the target* 0, *then* $0 \in fr[K(t^*)]$. *Where $fr[K(t^*)]$ denotes the frontier of $K(t^*)$.*

Proof of the remak (9). Consider $t_1 < t^* < t_2$ and suppose that $0 \notin K(t)$ for all $t_1 < t < t^*$, and $0 \in K(t)$ for all $t^* \leq t \leq t_3$. If there exists $\epsilon > 0$ such that for each $t_2 \geq t \geq t^*$ the family of open balls $\{B(0, r)\}$ for all $r \leq \epsilon$ are in the interior of $K(t)$ the function $dist[K(t), K(t)]$ is not continuous. Then for all $\epsilon > 0$ there exists t_ϵ and $r(t_\epsilon) \leq \epsilon$ such that $B(0, r(t_\epsilon)) \cap Fr[K(t_\epsilon)] \neq \emptyset$. So, it is possible to choose a convergent subsequence $\{t_n\} \in [t^* t_2]$ such that $r(t_n) \to 0$ let $\tilde{t}$ be the limit of $\{t_n\}$. Since $0 \in K(t)$ for all $t^* \leq t \leq t_3$, then $0 \in Fr[K(\tilde{t})]$. Suppose now that $0 \in intK(t)$. Repeating successively the previous reasoning we found $t^* \leq \tilde{t_1} < \tilde{t}$ such that $0 \in Fr[K(\tilde{t_1}]$. So repeating successively this reasoning we can construct a convergent sequence to t^* of $\{\tilde{t_n}\}$. So, for the continuity of the distance function $0 \in Fr[K(t^*)]$.

4 The linear control process

Consider an autonomous system

$$\dot{x} = A(t)x + B(t)u + v(t), \tag{2}$$

where:

1. $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times m$ matrix, and $v(t)$ is a column n-vector of real measurable functions on $-\infty < t < \infty$.

2. The norms $|A(t)|, |B(t)|$, and $|v(t)|$ are integrable on each compact interval of time t.

3. A controller $u(t)$ is a real bounded measurable m-vector on some interval $\mathcal{I}$.

A response or solution $x(t)$ is a real absolutely continuous n-vector on $\mathcal{I}$ which satisfies (2),

$$x(t) = \Phi(t)x_0 + \Phi(t)\int_{t_0}^{t_1} \Phi(s)^{-1}[B(s)u(s) + v(s)]ds.$$

Here $\Phi(t)$ is the fundamental matrix solution of the homogeneous system $dotx = A(t)x$ with $\Phi(t_0) = I$. If $A(t) = A$ is constant, $\Phi(t) = e^{A(t-t_0)}$.

4.1 Controllable set's properties.

Consider a linear control process, whose target state is $\mathcal{T}(t) \equiv 0$. Control variables are restricted to the restraint set Ω which is compact although it does not need to be convex, but in order to get easier proves we assume in some cases its convexity. Some of the properties next properties are true in more general cases than the linear one.

- 1. If $x_0 \in \mathcal{C}$ and y is a point in the trajectory from $\mathbf{x_0}$ to $\mathcal{T}(t)$, then $y \in \mathcal{C}$. *This means that the whole of the successful path from x_0 to the target lies in the controllable set.*
- 2. $\mathcal{C}$ is arc wise connected. If x_0 and y_0 are in $\mathcal{C}$ there is a path from each point to the origin lying wholly in $\mathcal{C}$. This proves that $\mathcal{C}$ is not composed of a number of disjoint parts.
- 3. If $t_1 < t_2$, then $\mathcal{C}(t_1) \subseteq \mathcal{C}(t_2)$.
- 4. $\mathcal{C}$ is open if and only if $0 \in int(\mathcal{C})$.
- 5. In the linear case, $\mathcal{C}(t_1)$ and $\mathcal{C}$ are symmetric and convex.

Remark 10 *Some of the properties previously enunciated are verified under conditions more general than the linearity of the process*

- *The property 1 is also verified for nonlinear autonomous process, $\dot{x} = f(x, u)$ whenever, for example, f is a continuous function.*
- *Property 2 is verified if $f(0, 0) = 0$; this ensures that once the target is reached, it is possible to remain there by switching off all the controls. enditemize*

Let $x(t)$ be denote the response for the control system with restraint set Ω, initial point x_0 and all controllers $u(t) \subseteq \Omega$. *Proof of the claims.*

1. Suppose that the trajectory is given by $x(t)$, with the control $u(t)$ Assume that at $t = t_1, x(t_1) = 0$. Suppose that $\bar{y} = x(\tau)$. consider now the control

$$v(t) = u(t + \tau)$$

This is an admissible control. Thus $\bar{y} \in \mathcal{C}(t_1 - \tau)$ and then $\bar{y} \in \mathcal{C}$

2. Suppose that $x_0 \in \mathcal{C}$ and $y_0 \in \mathcal{C}$, with $x(t_1) = 0$ and $y(t_2) = 0$. Consider the arc conformed by this trajectory, it is a path that between these two points, then $\mathcal{C}$ is a connected set.

3. Suppose $t_1 \leq t_2$ and consider $x_0 \in \mathcal{C}(t_1)$ with control $u(t)$. Define

$$v(t) = \begin{cases} u(t) & t_1 \leq t \\ 0 & t_1 \leq t \leq t_2 \end{cases}$$

Since $f(0,0) = 0, x(t) = 0\ \forall t \geq t_1$. So $x_0 \in \mathcal{C}(t_2)$

4. If $0 \in int[\mathcal{C}]$, there exists $\epsilon > 0$ such that $B(0,r) \subset \mathcal{C}$ for all $r < \epsilon$ where $B(0,r)$ is the open ball of radio equal to r. Let $\phi(t, t_0, x_0)$ be the solution of the dynamical process $\dot{x} = f(x,u)$ with $x(0) = x_0$ where $f(x,u) \in C^1$. It is well know that $\phi(t,t_0,x_0)$ is of class C^1 in x_0. We will prove that there exists $\epsilon > 0$ such that for all $r \leq \epsilon, B(x_0, r) \in int[\mathcal{C}]$. Let $y_0 \in B(x_0, \bar{r})$. Let $y(t)$ be the solution of $\dot{x} = f(x,u)$ with $y(0) = y_0$. Then for $\bar{r}$ small enough $y(t_1) \in B(0,r)$. So, there exists an admissible control v such that steer $y(t_1)$ to the origin at time t_2. Then it is possible to steer y_0 to the origin in time $t = t_1 + t_2$ for this consider the control

$$\tilde{v}(t) = \begin{cases} u(t) & t \leq t_1 \\ v(t - t_1) & t_1 \leq t \leq t_1 + t_2 \end{cases}$$

So $y_0 \in \mathcal{C}$.

Since 0 is controllable, the reciprocal part is straightforward.

5. The convexity of $\mathcal{C}$ is obtained from the following equality:

$$cx_1 + (1-c)x_2 = -\int_0^{t_1} \Phi(s)^{-1} B[cu_1 + (1-c)u_2] d\tau. \tag{3}$$

Recall that $x(t) = \Phi(t)\left[x_0 + \int_0^t \Phi(s)^{-1} Bu(\tau) + v(\tau) d\tau\right]$ is the solution of $\dot{x} = A(t)x + B(t)u + v(t)$ with $x_0 = x(0)$. So, $x_0 \in \mathcal{C}(t_1)$ if $x(t_1) = 0$, then x_0 is controllable if and only if

$$x_0 = -\int_0^{t_1} \Phi(s)^{-1} Bu(\tau) d\tau.$$

So, if $x_1, x_2 \in \mathcal{C}(t_1)$ with controllers u_1 and u_2 its convex combination also belongs to $\mathcal{C}(t_1)$ with the control $v = cu_1 + (1-c)u_2$ that is an admissible controller.

Exercise 1 *Consider a linear control process:*

1. *Show that the controllable set is symmetric.*

2. *Let $t_1 \leq t_2$. Show that if $x_0 \in \mathcal{C}(t_1)$ and $y_0 \in \mathcal{C}(t_2)$ then $z_0 = \alpha x_0 + (1-\alpha) y_0 \in C(t_2)$.*

4.2 The attainable set of a linear control

The main characteristics of the attainable set for linear control problems are summarizing in the next theorem. We will give an idea of the demonstration of the theorem and we left for the reader the task of completing it.

Theorem 11 *The attainable set of a linear control process $\dot{x} = A(t)x + B(t)u + v(t)$, with compact*[2] *convex restraint set Ω, initial state x_0 and controllers $u(t)$ on $t_0 \leq t \leq t_1$ is compact, convex and varies continuously with t_1 on $t_1 \geq t_0$.*

The proof of this claim follows from the properties of the variation of parameters formula

$$x(t) = \Phi(t)x_0 + \Phi(t)\int_{t_0}^{t_1} \Phi(s)^{-1}[B(s)u(s) + v(s)]ds.$$

See [Lee, E.; Markus, L.] 69.

Remark 12 *It is important to notice that if $K(t)$ varies continuously and if $P \in int(K(t_1))$ then, there exists $\delta > 0$, such that $P \in int(K(t_2))$ for all $|t_2 - t_1| < \delta$.*

4.3 Examples

To illustrate these ideas let us consider the following simple examples.

Example 13 *Consider the state equation*

$$\dot{x} = Ax + Bu,$$

where $A(n \times n)$ and $B(n \times m)$ are constant matrices.

[2] In order to show the compactness of this set, consider a sequence of admissible controllers $u_n(t)$ and its responses $x_n(t)$. Since Ω is a weakly compact set there exists a subsequence u_{n_i} which converges weakly to $\bar{u}(t) \in \Omega$. So, x_{n_i} the correspondent subsequence of responses converges to $\bar{x}(t) = \Phi(t)\left[x_0 + \int_0^t \Phi(s)^{-1}B\bar{u}(\tau) + v(\tau)d\tau\right]$

It follows that

$$x(t) = exp(At)\left(x_0 + \int_0^t exp(-A\tau)Bu(\tau)d\tau\right).$$

We see that $x_0 \in \mathcal{C}(t_1)$ if and only if there exists a control admissible, $u \in \Omega$ such that:

$$x_0 = -\int_0^{t_1} exp(-A\tau)Bu(\tau)d\tau.$$

- A point x_1 is in the attainable set $\mathcal{K}(t_1, x_0)$ when:

$$x_1 = exp(At_1)\left(x_0 + \int_0^{t_1} exp(-A\tau)Bu(\tau)d\tau\right), \tag{4}$$

  for some $u \in \Omega$.

If we define the *time reversed system* where the state equation is:

$$\dot{x} = -Ax - Bu.$$

- we can define $\mathcal{C}(t_1, x_1)$ as the set of points that are controllable to x_1 in time t_1, so x_0 will belong to this set if:

$$x_0 = exp(-At_1)\left(x_1 - \int_0^{t_1} exp(-A\tau)Bu(\tau)d\tau\right), \tag{5}$$

  for some control $u \in \Omega$.

There is obviously a reciprocal relationship between the two sets: if $x_1 \in \mathcal{K}(t_1, x_0)$, then $x_0 \in \mathcal{C}(t_1, x_1)$. so the controllable set for the time reversed system is equal to the attainable set for the original system. Analogously for the attainable set.

Remark 14 *Although we showed the reciprocity for the linear system, as it is easy to see, the same holds for the Non Linear Autonomous System, but not for non autonomous systems.*

Example 15 *Let us consider a very simple one-dimensional system,*

$$\dot{x} = x + u, \;\; x_0 = \frac{1}{2}, \;\; |u| \leq 1.$$

Points in the reachable set at time t_1 are given by

$$x_1 = exp(t_1)\left(\frac{1}{2} + \int_0^{t_1} exp(-\tau)u(\tau)d\tau\right)$$

and the bounds on the possible values u can take show that $\mathcal{R}(t_1, \frac{1}{2})$ is the closed interval from:

$$1 - \frac{1}{2}exp(t_1) \ \ to \ \ \frac{3}{2}exp(t_1) - 1.$$

Similarly, points in the reachable set for the time-reversed system are given by:

$$x_1 = exp(-t_1)\left(\frac{1}{2} - \int_0^{t_1} exp(\tau)u(\tau)d\tau\right),$$

so this reachable set is the closed interval from:

$$\frac{3}{2}exp(-t_1) - 1 \ \ to \ \ 1 - \frac{1}{2}exp(-t_1),$$

this is the controllable set to the point $\frac{1}{2}$ for the original system in the time t_1.

Example 16 *Let us now consider the case where: A and B are given by:*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the state equations are:

$$\dot{x_1} = x_1 + u_1, \ \ \dot{x_2} = x_2 + u_1,$$

and $u_1 \in \mathcal{U}_b$ that is, $-1 \leq u_1 \leq 1$, from the general result we see that $x \in \mathcal{C}(t_1)$ if:

$$x_1 = -\int_0^{t_1} exp(-\tau)u_1 d\tau = x_2,$$

since $exp(At) = exp(t)\mathbf{I}$.

Because $|u_1| \leq 1$, *then* $|x_1| \leq \int_0^{t_1} exp(-\tau)u_1 d\tau = 1 - exp(-t_1)$, and equality is possible, hence

$$\mathcal{C}(t_1) = \{x_1 = x_2, \ \ |x_1| = 1 - exp(-t_1)\},$$

and $\mathcal{C}$ is the open interval:

$$\mathcal{C}(t) = \{x_1 = x_2, \ \ |x_1| < 1\}.$$

As part of $R^2, int(\mathcal{C})$ is empty and $\mathcal{C}$ is not open.

It is impossible to control initial states that do not lie on this interval. Next we shall see the necessary conditions for a system to be completely controllable.

4.4 The controllability matrix:

Consider the autonomous linear control process

$$\dot{x} = Ax + Bu,$$

for real an constant matrices A and B. Here we have assumed that 0 is an equilibrium for the free system $u \equiv 0$, more general autonomous linear systems can be considered in this form by translation of the coordinates in both x and u.

The n rows and mn columns matrix:

$$M = [B \ AB \ A^2B...A^{n-1}B]$$

is the controllability matrix.

This matrix for the example 4.3 is given by:

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and for the example 1 in section 2

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The results that we will establish will depend on the rank of this matrix and on the characteristic of the eigenvalues of A. Observe that in the first case $n = 2$ and the rank of A $rank[A] < 2$. Whereas in the second $rank[A] = n$.

It is clear that the system cannot be completely controllable when $0 \notin int(\mathcal{C})$, for there would be points close to the origin that are not controllable.

- 1) $0 \in int(\mathcal{C})$ *if and only if rank* $M = n$ This establishes that if the rank of M is less than n, $\mathcal{C}$ lies in a hyperplane in R^n and the systems definitely are not completely controllable. However, if the rank is equal to n, the system may or may not be completely controllable. To obtain it we need additional conditions, these are established in the following items.

- 2)*If rank* $M = n$, *and* $u \in \Omega_u$, then $\mathcal{C} = R^n$, where Ω_u is the set of not bounded integrable controllers.

- 3) *If rank* $M = n$, and *Re* $\lambda_i < 0$ *for each eigenvalue* λ_i of A, $\mathcal{C} = R^n$ with $u \in \Omega_b$.

- 4)*If rank* $M = n$, and *Re* $\lambda_i > 0$ *for at least one eigenvalue* of A, $\mathcal{C} \neq R^n$.

- 5) *If rank* $M = n$, and *Re* $\lambda_i \leq 0$ *for each eigenvalue* λ_i of A, $\mathcal{C} = R^n$ with $u \in \Omega_b$.

Proof of the claims:

(1) Suppose that *rank* $M < n$, then there exists a constant row vector $y \in R^n$ such that $v[B\ AB\ ...\ A^{n-1}B] = 0$. By the Hamilton-Cayley theorem, each matrix A satisfies its own characteristic equation:

$$A^n = c_1 A^{n-1} + ... + c_n I$$

for ceratin real numbers $c_1, ..., c_n$. So $yA^kB = 0$ for all integer k. Then $yexp(-A\tau)B = 0$, hence for all $x_0 \in \mathcal{C}(t_1)$,

$$yx_0 = -\int_0^{t_1} yexp\,(-A\tau)\,Bu(\tau)d\tau = 0.$$

Therefore $\mathcal{C}(t_1)$ lies in an hyperplane with normal $\mathbf{y}$ for all t_1 and then $\mathcal{C}$ lies in the same hyperplane. Thus $0 \notin int[\mathcal{C}]$.

Now suppose that $0 \notin int\mathcal{C}$. This means that $0 \notin int[\mathcal{C}](t_1)$ for all $t_1 > 0$. But $0 \in \mathcal{C}(t_1)$ and since $\mathcal{C}(t_1)$ is a convex set, there exists an hyperplane trough 0 supporting $\mathcal{C}(t_1)$. If $b(t_1)$ is the normal to this hyperplane it follows that $b(t_1)x_0 \leq 0$, for all $x_0 \in \mathcal{C}(t_1)$ since $\mathcal{C}(t_1)$ is symmetric, it follows that $-x_0 \in \mathcal{C}(t_1)$, then $b(t_1)x_0 = 0$. We obtain that

$$\int_0^{t_1} b(t_1)exp\,(-A\tau)\,Bud\tau = 0,$$

for all $u \in \Delta$. The vanishing of the integral implies that

$$b(t_1)exp\,(-A\tau)\,B = 0,\ \ \forall\, 0 \leq \tau \leq t_1.$$

If we put $\tau = 0$ then $b(t_1)B = 0$. Taking k derivatives, and setting $\tau = 0$, we obtain $b(t_1)A^kB = 0$. It follows that $b(t_1)$ is orthogonal to all the columns of $\mathbf{M}$. Thus the rank of $\mathbf{M}$ is less than n.

(2) We have established that 0 is an interior point of $\mathcal{C}$, so there exist an open ball $B_0(r) \subseteq \mathcal{C}$. Let x_0 an arbitrary point in R^n. Then $y_0 = cx_0$ where $c = \frac{1}{2}r/\|x_0\|$ is in the ball, then there exists a control $v \in \Omega_u$, and so with the control $u = \mathbf{v}/c\ \in \Omega_u$ we obtain that

$$x_0 = -\int_0^{t_1} exp\,(-A\tau)\,Bu(\tau)d\tau,$$

hence $x_0 \in \mathcal{C}$.

(3) Let an initial point $x_1 \in R^n$ be steered by the null control $u(t) \equiv 0$ until the response $x(t)$ approaches 0 and enters $\mathcal{C}$. But then $x(t)$ can be steered to the origin in a finite time. Hence $x_1 \in \mathcal{C}$ and $\mathcal{C} \in R^n$.

Note that we could not deduce this result directly from the asymptotic stability of the system, because the definition of controllable set requires that the target can be reached in a finite time.

(4) Suppose that λ is an eigenvalue of A and $Re\lambda > 0$, and let y be the associated eigenvector, then $y^t A = \lambda y^t$ so that $y^{tr} exp(-A\tau) = exp(-\lambda\tau) y^{tr}$ and

$$y^{tr} x_0 = -\int_0^{t_1} exp(-\lambda\tau) y^{tr} Bu(\tau) d\tau.$$

Because $u \in \Omega_b$ this integral is bounded by p, say, as $t_1 \rightarrow \infty$, and so $y^t x_0 < p$. The controllable points lie in a half space on one side of an hyperplane in R^n and so $\mathcal{C} \neq R^n$.

(5) Observe that, even in the case when $rank(A) = n$, $Re\ \lambda_i \leq 0$ *for each eigenvalue* λ_i of A does not imply local stability of the solution

Suppose that $\mathcal{C} \neq R^n$ then there is an hyperplane with normal b separating y and $\mathcal{C}$, such that

$$bx_0 \leq p \ \ for\, all \ \ x_0 \ \in \mathcal{C}, and \ \ by > p.$$

We needed to show that for t_1 sufficiently large and for some control $u \in \Omega_b$

$$bx_0 = -\int_0^{t_1} bexp(-A\tau) Bu(\tau) d\tau > p$$

which will be a contradiction. Define $z(t) = bexp(-A\tau)B$. Because rank $\mathbf{M} = n,\ z(t) \neq 0,\ for\ 0 \leq t \leq t_1$, and we choose $u_i(t) = -sgnz_i(t)$ so that

$$bx_0 = -\int_0^{t_1} |v(t)| dt. \tag{6}$$

Where $v(t) = bexp(-A\tau)Bu(t)$. Each component of v is a combination of terms of the general form $q(t)exp(-\lambda_i t)$, where q is a polynomial and λ_i is an eigenvalue of A (see appendix of this section). It is clear that (6 can be made arbitrarily large for t_1 large. It grows exponentially if the terms with $Re(\lambda_i) < 0$ are present ore they are polynomials in t if $Re(\lambda_i) = 0$.

To clarify this point let us consider the following example:

Example 17 *Let $\dot{x} = Ax + Bu$ be the two components system:*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The eigenvalue of A are: $\lambda_1 = 0, \;\; \lambda_2 = -2$.

$$M = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

whose rank is 2.

$$exp(-At) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\; - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} t \;\; + \begin{bmatrix} 0 & -2 \\ 0 & 4 \end{bmatrix} \frac{t^2}{2!} - \begin{bmatrix} 0 & 4 \\ 0 & -8 \end{bmatrix} \frac{t^3}{3!} + ... = \begin{bmatrix} 1 & \frac{exp(-2t)-1}{2} \\ 0 & exp(2t)) \end{bmatrix}$$

$$z(t) = [b_1, b_2] \begin{bmatrix} 1 & \frac{-1-exp(2t)}{2} \\ 0 & exp(2t)) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b_1 \frac{-exp(2t) - 1}{2} + b_2 exp(2t).$$

finally we obtain : $|v(t)| = |b_1 \frac{-1-exp(2t)}{2} + b_2 exp(2t)|$.

Exercise 2 *Consider examples 1 and 2 in section 2.*

1. *Show that every initial point can be steered to the origin.*

2. *Compute the controllable set $C(t_1)$ at time $t_1 = 1$.*

3. *Compute the attainable set $K(t_1)$ at time $t_1 = 1$.*

4.5 Appendix:Exponential matrices and Resolution of differential equations

From a square real or complex $n \times n$ matrix A we define the matrix:

$$expA = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ... + \frac{A^k}{k!} + ...$$

The convergence is defined by the convergence of each component.

The Jordan canonical structure. For every complex $n \times n$ matrix A, there is a nonsingular complex matrix P such that

$$PAP^{-1} = diag\,\{A_1, A_2, ...A_k\}$$

where each block

$$A_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix}$$

involves eigenvalue λ_1 of A.

The following equality is straightforward: $Pexp[A]P^{-1} = exp[PAP^{-1}]$.

If $A(t)$ is integrable on each compact subinterval of $\mathcal{I}$, then for given initial data $t_0 \in \mathcal{I}$, there exists a unique, absolutely continuous fundamental solution matrix $\Phi(t)$ (or $\Phi(t, t_0)$) on $\mathcal{I}$, with $\Phi(t_0) = I$. The solution of

$$\dot{x} = A(t)x, \ \ with \ \ x(t_0) = x_0$$

is

$$x(t) = \Phi(t)x_0.$$

If $A(t) = A$ is constant, then: $\Phi(t) = e^{A(t-t_0)}$.

Now consider the non-homogeneous linear differential system

$$\dot{x} = A(t)x + B(t)$$

for a given prescribed initial data $x(t_0) = x_0$, $b(t)$ is a n-column integrable vector. The solution $x(t)$ is given by the fundamental formula:

$$x(t) = \Phi(t)x_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}B(s)ds$$

Here $\Phi(t)$ is the fundamental matrix solution of the corresponding homogeneous system:

$$\dot{x} = A(t)x.$$

A direct calculation verifies that this formula yields the required solution.

5 The time-optimal control. A first approach.

The Balancing problem. A tightrope walker seeks to maintain an upright position; any deviation from the vertical will result in a catastrophic fall unless it is controlled, and the acrobat's objective is to regain the (unstable) equilibrium position as quickly as possible. The dynamic of this system is multidimensional. We will consider a simple one-dimensional model where: $\dot{x} = x + u$. If $u = 0$ the solution $x_1 = 0$ is possible, but any initial non zero value of x leads to the exponential growth of x.

Suppose that we choose an initial value $x_0 = c > 0$ and try to find a control $u(t)$ that will steer x_1 to 0 in the shortest possible time. Suppose that $\mathcal{U}_b = \{u : |u(t)| \leq 1\}$. Is easy to see that the solution of the state equation is:

$$x(t) = e^t\left(c + \int_0^t e^{-\tau}u(\tau)d\tau\right).$$

The reachable set from the initial point at time t_1 is the closed interval:

$$(c-1)exp(t_1)+1 \leq x_1 \leq (c+1)exp(t_1)-1.$$

The end points of this interval are reached by the application of control equal to -1 and $+1$ respectively, for $0 < t < t_1$.

- Provided that $c < 1$ this set contain the origin when $t_1 \geq -ln(1-c)$ and thus not contain the origin when $0 \leq t_1 < -ln(1-c)$.

  Hence the equilibrium can be regained in the optimal time $-ln(1-c)$by application of $u(t) = -1$.

- If $c > 1$ the initial state is not controllable.

Suppose we repeat the problem, but with initial value $x_0 = 2$ and with target $x_1 = 3$. The extremal control are $u_1(t) = 1$ and $u_1(t) = -1$, as before and the corresponding extremal trajectories are:

$$x(t) = 3e^t - 1 \ \ when \ \ u_1 = +1, \ \ x(t) = e^t + 1 \ \ when \ \ u_1 = -1$$

Both these trajectories reach at the target; the first at time $t_1 = ln\frac{4}{3}$ and the second at $t_1 = ln2$. We have two acceptable candidates for the optimal solution, if we must pick the one with the smaller value of t_1 thus $t_1^* = ln\frac{4}{3}, \ u^*(t) = +1$. The other extremal solution reaches the target as long as possible.

6 The Maximun Principle for linear processes

In this section we shall establish the maximal principle for a linear process

$$(\mathcal{L}) \qquad \dot{x} = A(t)x + B(t)u + v(t)$$

where the coefficient matrices $A(t), B(t)$, and $v(t)$ are integrable in every finite interval.

This principle characterizes the optimal controller as an extremal controller that is: *If a control u^* is optimal in the sense that the associated response $x^*(t)$ reaches the target in minimal time, then $u^*(t)$ is an extremal control, i.e $x^*(t^*) \in fr[K(t^*)]$, where t^* is the first time that $\mathcal{T}(t) \cap K(t) \neq \emptyset$.*

Next we shall prove the fundamental existence and uniqueness theorems for optimal controllers of linear process

6.1 Extremal controllers and the maximal principle

Definition 18 *Let $u(t) \subseteq \Omega$ on $t_0 \leq t \leq t_1$ be a controllers for the linear process $(\mathcal{L})$ with initial state x_0 at time t_0. If the corresponding response $x(t)$ has an endpoint $x(t_1)$ in the boundary $fr(K(t_1))$ of the set os attainability $K(t_1)$, then $u(t)$ is called an* **extremal controller** *and $x(t)$ is an extremal response.*

In order to express the extremal condition analytically we shall use the adjoint linear differential system

$$(\mathcal{A}) \qquad \dot{\eta} = -\eta A(t)$$

to the linear differential system: $\dot{x} = A(t)x$.

Here $\eta(t)$ is an $n-$row vector and every solution of it, is of the form $\eta(t) = \eta_0 \Phi(t)^{-1}$, where η_0 is a constant vector and $\Phi(t)$ is the fundamental matrix solution of $\dot{x} = A(t)x$, with $\Phi(t_0) = I$. This claim can be verified by direct differentiation. In the case $A(t) = A$ where $A(t)$ is a constant matrix, it follows that $\eta(t) = \eta_0 e^{-(t-t_0)A}$.

The following theorem, is the principal analytical device in the theory of time-optimal control problem for linear processes, and it is equivalent to the Pontriaguin maximal principle for this case.

Theorem 19 *Consider the linear process in R^n*

$$\dot{x} = A(t)x + B(t)u + v(t) \tag{7}$$

with u in a compact restraint set Ω and initial state x_0 at time t_0. A controller $u(t) \in \Omega$ on $t_0 \leq t \leq t_1$ is extremal if and only if there exists a nontrivial solution $\eta(t)$ of

$$\dot{\eta} = -\eta A(t)$$

such that

$$\eta(t)B(t)u(t) = \max_{u \in \Omega} \eta(t)B(t)u,$$

for almost all t on $t_0 \leq t \leq t_1$.

Proof: Assume that $u(t)$ on $t_0 \leq t \leq t_1$ is extremal and so steers x_0 to $x(t_1) \in fr(K(t_1))$ where $k(t_1)$ is the attainable set on t_1 by the response:

$$x(t) = \Phi(t)x_0 + \Phi(t) \int_{t_0}^{t} \Phi(s)^{-1}[B(s)u(s) + v(s)]ds.$$

Since $K(t_1)$ is compact and convex, there is a supporting hyperplane π to $K(t_1)$ at the boundary point $x(t_1)$. Let $\eta(t_1)$ denote the unit normal vector to π at $x(t_1)$ in the direction opposite to $K(t_1)$. Define the nontrivial adjoint response

$$\eta(t) = \eta_0 \Phi(t)^{-1} \;\; with \;\; \eta(t_1) = \eta_0 \Phi(t_1)^{-1},$$

then compute the inner product of $\eta(t)$ and $x(t)$,

$$\eta(t)x(t) = \eta_0 x_0 + \int_{t_0}^{t} \eta(s)[B(s)u(s) + v(s)]ds.$$

Now suppose that:

$$\eta(t)B(t)u(t) < \max_{u \in \Omega} \eta(t)B(t)u$$

Define a controller $\bar{u}(t) \in \Omega$ by:

$$\eta(t)B(t)\bar{u}(t) = \max_{u \in \Omega} \eta(t)B(t)u$$

We find that

$$\eta(t_1)x(t_1) < \eta(t_1)\bar{x}(t_1).$$

But this inequality contradicts the construction of $\eta(t_1)$. Therefore we conclude that:

$$\eta(t)B(t)u(t) = \max_{u \in \Omega} \eta(t)B(t)u \;\; a.e. \; t_0 \leq t \leq t_1.$$

Conversely: assume that for some non trivial adjoint response: $\eta(t) = \eta_0 \Phi(t)^{-1}$, the control $u(t)$ satisfies:

$$\eta(t)B(t)u(t) = \max_{u \in \Omega} \eta(t)B(t)u$$

almost everywhere in $t_0 \leq t \leq t_1$. We must show that the corresponding response $x(t)$ ends at a boundary point of $K(t_1)$. Suppose that $x(t_1)$ lies in the interior of $K(t_1)$ then there exists a point $\bar{x}(t_1) \in K(t_1)$ such that $\eta(t_1)x(t_1) < \eta(t_1)\bar{x}(t_1)$. Let $\bar{u}(t)$ be the control which yield the response $\bar{x}(t)$, our hypothesis states:

$$\eta(t)B(t)\bar{u}(t) \leq \eta(t)B(t)u(t) = \max_{u \in \Omega} \eta(t)B(t)u$$

almost everywhere in $t_0 \leq t \leq t_1$. Computing we obtain:

$$\eta(t_1)\bar{x}(t_1) \leq \eta(t_1)x(t_1),$$

which is a contradiction, hence $x(t_1) \in fr(K(t_1))$, as required.[.]

Remark 20 *This theorem states that the response $x(t)$ leading to a boundary point of $K(t_1)$, go in the appropriate direction at the greatest possible speed compatible with the restraint Ω.*

The geometry of the solution of $\mathcal{L}$ is given by the following corollary, hose demonstration we left for the reader.

Corollary 21 *If $u(t) \in \Omega$ is an extremal controller for the process $\mathcal{L}$, then $u(t)$ is extremal on each subinterval $t_0 \leq \tau \leq t_1$, that is $x(\tau) \in K(\tau)$. Furthermore, $\eta(\tau)$ is an exterior normal to a supporting hyperplane to $K(\tau)$ at $x(\tau)$*

6.2 Extremal control for autonomous linear with constant coefficients processes

Consider the autonomous linear process in R^n :

$$\dot{x} = Ax + Bu,$$

where A and B are constant matrices.

From theorem (19) it follows that the control $u(t)$ is extremal if and only if there exists a non-null vector $\eta(t)$ such that, $\eta(t)B(t)u(t) = \max_{v(t)\in\Omega} \eta(t)B(t)v(t)$ for all t such that $0 \leq t \leq t_1$.

In the linear case we have

$$\eta(t) = \eta_0[e^{-At}].$$

Observe that $\eta(t)$ is a not trivial solution of the system $\dot{\eta} = -\eta A$.

So

$$u_i(t) = sgn[\eta_0 e^{-At} B_i].$$

When $[\eta_0 exp\,(-At)\,B]_i$ is zero the component u_i of an extremal control is not determined, in this case, since de function is analytical, there are only two possibilities:

- It has a finite number of zeros and the optimal control is uniquely determined except at a discrete number of moments, and the trajectory is not affected if for instance we assign +1 to the control in these points.

  Therefore there are only a finite number of switches for the extremal controller $u_i(t)$. So we obtain a *bang-bang* solution everywhere.

- In the second case $[\eta_0 exp\,(-At)\,B]_i$ is identically zero, therefore u_i is no determined.

Remark 22 *Recall that a function is called analytical in a if there exists an absolutely convergent power-series expansion about each point in a neighborhood of a.*

1. *If f is an analytical function in a and is not identically zero, then there exists some derivative $f^{(r)}(a) \neq 0$. Observe that the function $f(x) = e^{-\frac{1}{x^2}}$ $\forall\, x \neq 0$ and $f(0) = 0$, is not analytical function in* 0.

2. *If f is an analytical function and $f^r(a) = 0$, $\forall r$ then it is identically zero function.*

3. *If f and g are analytical functions such that, $f(x) = g(x)$ in a set which has an accumulation point, then $f(x) \equiv g(x)$. So if $f(x) = 0$ in a set with an accumulation point, then is identically zero.*

6.3 The time-optimal controllers and the maximun principle (TOP)

In this section we shall establish that the maximal principle, characterizes the optimal controller as an extremal controller. We shall study the minimal time-optimal control problem for the linear process

$$\dot{x} = A(t)x + B(t)u + v(t)$$

where the target $G(t)$ is a continuously varying nonempty compact set on $\tau_0 \leq t \leq \tau_1$, the coefficient matrices are integrable on every finite interval.

- (I) *The existence of the optimal solution for the linear autonomous case.*

Theorem 23 *Consider the linear process in R^n*

$$\dot{x} = A(t)x + B(t)u + v(t) \tag{8}$$

with u in a compact restraint set Ω and initial state x_0 at time t_0, and continuously varying compact target set $G(t)$ on $\tau_0 \leq t \leq \tau_1$. If there exists a controller $u(t) \subset \Omega$ on $\tau_0 \leq t \leq \tau_1$, steering x_0 to $G(t_1)$, then there exists a minimal time optimal controller *$u^*(t) \subset \Omega$ on τ_0‘$t \leq t^* \leq \tau_1$, steering x_0 to $G(t^*)$.*

Proof: Define t^* as the greatest lower bound of all times t_1 such that $K(t_1)$ meets $G(t_1)$. By the continuous dependence of $K(t_1)$ and $G(t_1)$ on t_1 the set of times for which $K(t_1)$ meets $G(t_1)$ is a closed set on R^1. Hence t^* on $\tau_0 \leq t^* \leq \tau_1$, is the first or minimal time at which $K(t)$ meets $G(t)$. Let $u^*(t) \in \Omega$ on $\tau_0 \leq t \leq t^*$, be any controller steering x_0 to $K(t^*) \cap G(t^*)$. Then $u^*(t)$ is an optimal controller as required.

- (II)*How can the optimal time and its associated control be founded?*

In the following theorem we shall show shows that in the case of a linear process,

(1) The associate response $x^*(t)$ to the optimal controller$u^*(t)$ satisfy $x^*(t^*) \in fr[K(t^*)]$. Thus and optimal control is an extremal control.

(2) Then, to obtain u^* we use the maximal principle:

Theorem 24 *Consider the linear process in R^n*

$$\dot{x} = A(t)x + B(t)u + v(t) \tag{9}$$

with u in a compact restraint set Ω and initial state x_0 at time t_0, and continuously varying compact target set $G(t)$ on $\tau_0 \leq t \leq \tau_1$. Let $u^*(t) \in \Omega$ on $\tau_0 \leq t \leq t^*$ be a minimal time optimal controller with response $x^*(t)$ steering x_0 to $G(t^*)$.

(1) Then $u^*(t)$ is extremal, that is

$$m(t) = \max_{u\in\Omega} \eta(t)B(t)u = \eta(t)B(t)u^*(t),$$

here $\eta(t)$ is a nontrivial solution of the adjoint system $\dot{x}(t) = -\eta A(t)$, and $\eta(t^*)$ is an outwards unit normal to a supporting hyperplane at $x(t^*) \in fr(K(t^*))$.

(2) Furthermore if

$$M(t) = \max_{u\in\Omega} \eta(t)[A(t)x^*(t) + B(t)u + v(t)] = \eta(t)[A(t)x^*(t) + B(t)u^*(t) + v(t)]$$

and $G(t) = G$ is constant, the normal $\eta(t^*)$ can be selected so that $M(t^*) \geq 0$.

(3) If in addition, $G(t)$ is convex, then $\eta(t^*)$ can also be selected to satisfy the transversality condition, namely $\eta(t^*)$ is normal to a common supporting hyperplane separating $K(t^*)$ and G.

(4) For the autonomous linear process in R^n

$$\dot{x} = Ax + Bu + v$$

$M(t)$ is constant on $\tau_0 \leq t \leq \tau_1$.

Proof:

1. The associate response x^* at any $t \leq t^*$ is on the $fr(K(t)$. Suppose that $x^*(\bar{t})$ belong to the interior of $K(\bar{t})$, then for $t' < \bar{t}$ sufficiently close of $\bar{t}$, since $K(t)$ is continuous, there exists a neighborhood of $x^*(\bar{t})$ such that meet $k(t')$ then $x^*(\bar{t})$ is reached in $t' < \bar{t}$ this contradicts the optimality of u^*. Since the response endpoint $x^*(t^*)$ must lie on the boundary $fr[K(t^*]$, the optimal controller $u^*(t)$ is extremal. So, item (1) follows.

2. Now let $\mathcal{T}(t) = G$ be a constant nonempty compact set in R^n. At each instant of time $t_1 \leq t^*$ there exists and midway hyperplane $\tilde{\pi}(t_1)$ perpendicular of the shortest chord from $K(t)$ to $x^*(t^*)$. Choose a sequence of times

$$\tau_0 \leq t_1 < t_1' < t_2 < t_2' < ... < t^*.$$

At some time t_h' on $\tau_0 \leq t_h \leq t_h' \leq t^*$ the velocity $\dot{x}^*(t_h) = A(t_h)x^*(t_h) + B(t_h)u^*(t_h) + v(t_h)$ must have a positive component along the unit normal $\bar{\eta}(t_h')$ to $\bar{\pi}(t_h')$ directed out of the half space containing $K(t_1)$. So,

$$\bar{\eta}(t_h')[A(t_h)x^*(t_h) + B(t_h)u^*(t_h) + v(t_h)] \geq 0.$$

Taking limits when $h \to \infty$ and using the compactness of Ω it follows from the continuity of $A(t), B(t), v/t)$ and $x^*(t)$ that

$$\bar{\eta}(t^*)[A(t^*)x^*(t^*) + B(t^*)u^*(t^*) + v(t^*)] \geq 0.$$

So $M(t^*) \geq 0$ as required.

3. If G is a convex target, then we repeat the above argument, choosing the shortest chord from G to $K(t)$. Then the limit hyperplane $\pi(t^*)$ separating $K(t^*)$ and G with the unit normal $\eta(t^*)$ satisfyng the required transversality condition.

4. It can be proved that $M(T)$ is absolutely continuous and has a derivative almost always. Thus for $t_2 > t_1$ we have:

$$\frac{M(t_1) - M(t_2)}{t_2 - t_1} \geq \frac{\eta(t_2)\,[Ax(t_2) + Bu(t_1) + v] - \eta(t_1)\,[Ax(t_1) + Bu(t_1) + v]}{t_2 - t_1}.$$

Upon adding and subtracting $\eta(t_2)Ax(t_1)$ to the numerator and computing the limit $t_2 \to t_1$ we find $\frac{dM}{dt}(t_1) \geq 0$. On the other hand:

$$\frac{M(t_1) - M(t_2)}{t_2 - t_1} \leq \frac{\eta(t_2)\,[Ax(t_2) + Bu(t_2) + v] - \eta(t_1)\,[Ax(t_1) + Bu(t_2) + v]}{t_2 - t_1}.$$

And a similar calculation shows that $\frac{dM}{dt}(t_1) \leq 0$.

- (III) *The next problem is the problem of the uniqueness.* If conditions for uniqueness of the optimal control can be founded then, under these conditions the maximal principle is a necessary and sufficient condition for optimality. Under conditions of normality, the optimal controller is the unique extremal controller steering x_0 to the fixed target set G, and satisfying the transversality conditions. This problem is the goal of the next section.

Exercise 3 *Suppose that the dynamics of an economy is given by the following system of equations*

$$\dot{I} = v$$
$$\dot{k} = I - \theta k$$

here $0 \le v \le 1$ *is the control variable, and* $\theta = \delta + n$, *where* δ *is the depreciation rate of capital and* n *is the rate of growth of the population. Also suppose that a central planner wishes to maintain the per-capita capital constantly equal to* $\bar{k}$ *, as well as to correct any deviation of this objective in the smaller possible time. Suppose that in order to attempt this objective , the central planer uses the rate of variation of investment* v, *as a policy instrument.*

1. *Show that the problem is completely controllable.*

2. *For each pair* (k_0, I_0) *of initial conditions, find an optimal control.*

3. *Show that the transversality conditions are verified.*

Exercise 4 *Consider the autonomous linear process in* R^n $\dot{x} = Ax + Bu$, *with compact restraint set* $\Omega \subset R^m$, *initial state* x_0, *and the origin as the fixed target set. Assume that* $u = 0$ *lies in the interior of* Ω. *Suppose that the process is controllable, and* A *is stable, that is each eigenvalue* λ *of* A *satisfies* $Re[\lambda] < 0$. *Show that in these conditions, there exists a minimal time controller* $u^*(t) \in \Omega$ *on* $0 \le t \le t^*$, *steering* x_0 *to the origin at time* $t = t^*$.

Exercise 5 *If the target set* $\mathcal{T}$ *is compact, we know that there exists a minimal time* t^* *in which the target is reachable. There is an important difference with the case where the target is only one point* x_1. *We have proven only that the target is reached in a point of its border, but we do not know exactly where.*

Let t^* *be the minimal time for a linear optimization problem. Show that if* $K(t^*)$ *is strictly convex, and if* $u_1^*(t)$ *and* $u_2^*(t)$ *are optimal controllers, then* $x_1^*(t^*) = x_2^*(t^*)$ *where* $x_1^*(t)$ *and* $x_2^*(t)$ *are the respective associate optimal responses.*

6.4 Normality: condition for uniqueness.

If there exists just one maximal controller steering x_0 to the target then the maximal principle is sufficient, as well as necessary, for optimality. In this setion we analize conditions for uniquenes of the optimal controller.

Definition 25 *Consider the linear control process*

$$\dot{x} = A(t)x + B(t)u + v(t)$$

with restraint set Ω initial state x_0 at time t_0. The problem defined by the data $(\mathcal{L}, \Omega, x_0, t_0, t_1)$ is called **normal** *in case any two controllers $u_1(t)$ and $u_2(t)$ on $t_0 \leq t \leq t_1$, which steer x_0 to the same boundary point $P_1 \in Fr[K(t_1)]$, are equal almost everywhere.*

As we shall see in this section, this condition is strongly related with the geometry of the attainable set.

We start this section with and example where there are infinitely many optimal optimal controllers.

Example 26 *Consider the two dimensional system $\dot{x} = u$, $(n = m = 2)$. Then*

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad e^{-At} = I \quad u(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and $\eta_0 e^{-At} B = \eta_0 \equiv (\alpha, \beta)$. Thus

$$(*) \qquad u_1(t) = sgn\ \alpha \quad u_2(t) = sgn\ \beta.$$

If either of α, β is zero, then (*) does not tell us anything about the corresponding component of $u(t)$.

Consider the initial point $(-1, 0)$. This state can be steered to the target $(0, 0)$ in the time $t_1 = 1$. The time is optimal because we cover the shortest distance using maximun velocity. This is not a bang-bang control, and we must have in this case $\alpha > 0$, $\beta = 0$. Actually there are infinitely many optimal controls steering from $(-1, 0)$ to $(0, 0)$ e.g., $u_1(t) \equiv 1$ and:

$$u_2(t) = \begin{cases} 1, & 0 \leq t \leq a, \\ 0, & a < t < 1 - a \\ -1, & 1 - a \leq t \leq 1. \end{cases}$$

The idea is to burst $x_2(t)$, and then bring it back to zero. This control will be bang-bang in exactly one case $(a = \frac{1}{2})$:

Analytically, in this example the difficulty stems from the fact that $\eta_0 e^{-At} B$ has a component identically zero; *geometrically,* the problem is that $K(t, x_0)$ has flat spots on its boundary.

Exercise 6 *With $m = n = 2$, consider $\dot{x} = u$ with $x_0 = (-1, 0), \mathcal{T}(t) \equiv 0$.*

1. *Sketch the reachable cone RC, in (x_1, x_2, t)-space where*

$$RC = \{(t, x(t, x_0, u(.))t \geq 0, u(.) \in U_b\} = \cap_{t \geq 0} \{t\} \times K(t; x_0).$$

   Show that the attainable set $K(t; x_0) = \{(x(t, x_0, u(.))t \geq 0, u(.) \in U_b\}$ is always a square.

2. *Show that the control* $u_1(t) \equiv 1, u_2(t) = \phi(t)$, *with* $\phi(.)$ *any function such that* $\int_0^1 \phi(s)ds = 0$, *is time optimal.*

3. *Show that if* Q *is a corner of* $fr[K(t, x_0)]$, *the the control steering to each* Q *is unique, but the support hyperplane* P *and normal* h *are not unique.*

4. *Show that if* Q *is not a corner of* $fr[K(t, x_0)]$ *the support hyperplane* P *and normal* h *are unique, but the path to* Q *is not unique.*

Definition 27 *We say that a linear autonomous process has the* **normality condition** *if for each vector with constant coordinates,* $\eta_0 \neq 0$ *no component of* $\eta_0 e^{At} B$ *can vanish on a set of positive measure.*

Of course this definition is of little practical use. We will show that this definition is equivalent to the:

1. *Geometrical definition.* (LA) normal if and only if $k(t, x_0)$ is strictly convex for all t. Recall that $y \in K(t, x_0)$ f and only if: $y = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds$. for some $u(.) \in \Omega_b$. Show that the convexity of $K(t, x_0)$ is independents of x_0.

2. *Analytical definition.* Let b_i be the *ith* column of B and define the $n \times n$ matrix $\mathbf{M_i}$ by:

$$\mathbf{M_i} = [b_i, Ab_i, ..., A^{n-1}b_i].$$

   We say that a the problem is normal if $rank\mathbf{M}_i = n; \;\; for \;\; i = 1, 2, ..., m.$

   Clearly, if any one of these matrices has rank equal to n, $\mathbf{M}$ contains n independent columns and also has rank equal to n. So, The conditions for uniqueness are more stringent than the controllability.

In the following theorems we give explicitly verifiable hypothesis to uniqueness for autonomous linear processes with convex target.

Theorem 28 *If the process*

$$\dot{x} = Ax + Bu + v$$

is normal, and if there exists a successful control (steering x_0 *to 0) then there exists a unique time-optimal control, which is bang-bang and piecewise constant.*

Proof: The existence of the solution and the fact that any solution is bang-bang (in all coordinate) follows because the process is linear and uniqueness follows from the normality hypothesis [3]. Suppose that $u(t)$ and $v(t)$ were two distinct time-optimal bang-bang controls. Then as the process is linear $w(t) = \frac{1}{2}[u(t) + v(t)]$ would also be time optimal, but not bang-bang, this is a contradiction.

Theorem 29 *Consider the autonomous linear process in R^n*

$$\dot{x} = Ax + Bu + v$$

with compact and polyhedral convex restraint $\Omega \subset R^n$ and initial state $x_0 \in R^n$. If

1. *The normality condition holds*

2. *G is a fixed compact target set and*

3. *For each point $\bar{x} \in G$ there is a controller $\bar{u}(t)$ with response $\bar{x}(t) \in G, \ \forall t : 0 \le t < \infty$.*

Then any extremal controller $u_1(t) \in \Omega$ on $0 \le t \le t_1$, steering x_0 to G and satisfying the transversality condition, is equal to the unique optimal controller $u^*(t)$ almost everywhere, and $t_1 = t^*$.

Proof: Suppose that there exist distinct extremal controllers $u_1(t)$ and $u_2(t)$ such that:

$$\eta(t)Bu_1(t) = \eta(t)Bu_2(t) = \max_{\omega} \eta(t)Bu$$

almost every where on $[0, \tau_1]$, where $\eta(t) = \eta_0 e^{-At}$ and $u_1(t) \neq u_2(t)$ on a set S of positive duration on $[0, \tau_1]$. Now as Ω is a convex set the function $\eta(t)Bu$ assumes for each $t \in S$ its maximun in a edge e_t of Ω. Since Ω has only a finite number of edges, then there exists some positive time duration $S_1 \subset S$ where the function assumes its maximal value on a fixed edge e_1. Let w be a parallel vector to e_1. Since $\eta(t)Bu = \eta_0 e^{-At}Bu$, then $\eta_0 e^{-At}Bw = 0$ then taking derivatives we obtain

$$\eta_0 e^{-At} A^k Bw = 0, \quad k = 0, 1, ..., n-1.$$

Then the vectors $A^k Bw \quad k = 0, 1, ..., n-1$ are orthogonal to $\eta_0 e^{-At}$ an the linearly dependent. This contradicts the normality condition.

Conditions (2) and (3) imply that $t_1 = t^*$. Suppose that $t_1 > t^*$, condition (3) implies that G meets $K(t)$ for all $t > t^*$, then $K(t_1)$ meets G and thus, $K(t_1)$ can be separated from G. In this case this is impossible, so $t_1 = t^*$.

[3] As counterexample see example (26).

Thus: *any extremal controller satisfying the transversality condition, in particular the optimal controller* $u^*(t)$ *on* $\tau_0 \leq t \leq t^*$ *must be equal* $u_1(t)$ *almost everywhere on* $\tau_0 \leq t \leq t_1 = t^*$.

Example 30 *Uniqueness does no imply normality. [Macky, J.; Strauss, A.] 66*

Definition 31 *Let* x_0 *be given. We say that the* **response from** x_0 *to a point* y *in* $K(t^*, x_0)$ *is* **unique** *if every control which steers from* x_0 *to* y *in time* t^* *generates the same response function, i.e., if* $u(.)$ *and* $v(.)$ *are successful controls for* $0 \leq t \leq t^*$ *then* $x(t, x_0, u) = x(t, x_0, v)$ *on* $[0, t^*]$.

Corollary 32 *Suppose that* $y \in K(t^*, x_0)$. *Then the control steering from* x_0 *to* y *at time* t^* *is unique if and only if* y *is an extreme point of* $K(t^*, x_0)$.

Example 33 *In the balancing problem,(this problem was already considered above) we have a simple one-dimensional model :* $\dot{x} = x + u$. *If* $u = 0$ *the solution* $x_1 = 0$ *is possible, but any initial non zero value of* x *leads to the exponential growth of* x. *In this case* $exp(At) = e^{-t}$ *and* $B = 1$. *If* $\eta_0 = \alpha$, *then*

$$u_1 = sgn(\alpha e^{-t}).$$

As it is easy to see this is not a normal problem.

In this case there are no possibility of $\alpha = 0$, there are only two possibilities for α : either $\alpha > 0$ and $u_1 = +1\ \forall t$, or $\alpha < 0$ and $u_1 = -1\ \forall t$.

Example 34 The position problem. *Consider the problem of moving an object along a line with a coordinate* x_1. *The state of the system is determined by the position and the velocity of the object, we need another state variable* x_2.

$$\dot{x_1} = x_2, \ \ \dot{x_2} = u_1$$

It is a two dimensional problem with a single control: $n = 2, \ M = 1$. The state equation is given by the matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This is a normal problem.

Suppose that the target is $x_1 = 0$. Then from theorem (28) we know that there exists only one extremal controller steering x_0 to x_1 and it is optimal.

Since $A^2 = 0$ so:

$$exp\,(At) = \mathbf{I} + At = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

Since, the solution of the adjoint system has the form $\eta = \eta_0 e^{At}$, if we suppose $\eta_0 = (\alpha, \beta)$, from the maximal principle:

$$u_1(t) = sgn(\beta - \alpha t).$$

The linear function of t has at most one zero, so u_1 has at most one switch from $+1$ to -1 or from -1 to $+1$. Since α and β are unknown, we cannot be sure from the maximal principle if and when such a switch will occur. If the initial and final states are given by: $x(0) = (0, x_0)$ and $x(t_1) = (0, 0)$.

(a) Beginning with $u_1 = -1$ the initial section of the trajectory can be found by solving de differential system. The solution is:

$$x_2 = -t, \;\; x_1 = x_0 - \frac{1}{2}t^2. \tag{10}$$

- *In the space of phases the trajectory is given by the equation:*

$$x_1 = \frac{1}{2}x_2^2 + x_0.$$

- At some time t_1 we switch the control to the value $+1$. At this time the initial conditions of the system are given by: (10) with $t = t_1$.

- Then we solve the differential system with $u = +1$. The solutions of this system are:

$$x_1 = \frac{1}{2}t^2 - 2tt_2 + x_0 + t_2^2, \quad\; x_2 = t - 2t_2.$$

- *In the spaces of phases the trajectory is given by the equation:*

$$x_1 = \frac{1}{2}x_2^2 - t_2^2 + x_0.$$

- We now find values t_1 and t_2 such that:

$$t_1 - 2t_2 = 0, \;\; \frac{1}{2}t_1^2 - 2t_1t_2 + x_0 + t_2^2 = 0.$$

  These values are: $t_1 = (x_0)^{\frac{1}{2}}, \;\; t_2 = \frac{1}{2}t_1$.

(b) The extremal controls admits a second possibility, with start with $u_1 = 1$ and switch to $u_1 = -1$. It is easy to verify that such control leads to an increase in the value of of x_1 and when $x_2 = 0$ we are further from the target than when we started. This sequence of controls does not enable the target to be reached, and so can be dismissed.

So, the optimal time is $t^* = t_1 + t_2 = \frac{3}{2}(x_0)^{\frac{1}{2}}$, and the optimal control is given by:

$$u^*(t) = \begin{cases} -1 & t \leq (x_0)^{\frac{1}{2}} \\ 1 & t > (x_0)^{\frac{1}{2}} \end{cases}$$

6.5 The backing out of target technique.

- (1) Use the reversed time system:

$$\dot{x} = -Ax - Bu(t) - v$$
$$\dot{\eta} = \eta A,$$

  with initial conditions $x(0) \in fr(G)$ and $\eta(0)$ being an inward unit normal to a supporting hyperplane to $\mathcal{T}$ at $x(0)$. Define $u(t)$ by the maximal principle,

$$\eta(t)Bu(t) = \max_{u\in\Omega} \eta(t)Bu$$

- (2) Find the solution $x(t), u(t)$ passing trough the prescribed initial state x_0 at some time $t^* > 0$.

- (3) Reverse the time sense again and define:

$$x^*(t) = x(t^* - t) \ \ and \ \ \eta^*(t) = \eta(t^* - t) \ \ on \ \ 0 \le t \le t^*.$$

  Then $u^*(t)$ defined by:

$$\eta^*(t)Bu^*(t) = \max_{u\in\Omega} \eta^*(t)Bu$$

  is the optimal controller and $x^*(t)$ the corresponding optimal response.

Example 35 *Consider the autonomous control process in* R^2

$$\mathcal{L} \quad ; \quad \begin{array}{c} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -x_2 + u \end{array}$$

with restraint $\Omega : |u| < 1$ *in* R^1. *We wish to steering to the line* $x_1 = 0$. *This implies that* $\dot{x}_1(t) = 0$ *so* $x_2(t) = u(t)$ *then* $|x_2| \le 1$. *Thus the target is the set* $G = \{x_1 = 0, |x_2| \le 1\}$

We note that each point in R^2 can be steered in $\mathcal{T}$ by a non extremal controllers $u(t) = -x_2 e^{-2t}$, $t \ge 0$.

Using the coefficient matrices:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we can verify that the system is controllable and that the *normality condition for uniqueness of the optimal controller holds.*

From the controllability an normality of the system, it follows that each initial state in R^2 can be steered to $\mathcal{T}$ by an unique optimal control satisfying the transversality condition. Since

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

condition (1) of theorem (29) is verified. To see that condition (2) of this theorem follows, consider the control $\bar{u}(t) = \bar{x}_{20} e^{-2t}$ where $\bar{x} = (0, \bar{x_2 0})$ where $|x_{20}| \leq 1$.

Write the system and the adjoint system with time reversed

$$\begin{aligned} &\dot{x}_1 = -x_2 - u \quad and \;\; u = sgn(\eta_1 + \eta_2) \\ &\dot{x}_2 = x_2 - u \\ &\dot{\eta_1} = 0 \\ &\dot{\eta_2} = \eta_1 - \eta_2. \end{aligned}$$

Note that along a solution where $\dot{\eta_1} = 0$, it follows

$$\eta_1 + \eta_2 = c_1 + c_2 e^{-t}.$$

Then each extremal control has at most one switch on $0 \leq t < \infty$.

We examine all extremal controllers satisfying the transversality condition to obtain the switching locus W.

- For the time reversed system, take initial data: $x_1(0) = 0,\; |x_2(0)| < 1,\;\; \eta_1(0) = \pm 1,\;\; \eta_2(0) = 0$. Then $\eta_1 + \eta_2 = \pm 2 \mp e^{-t}$ so there are no switches.
- Use the value $u = -1$ and start from $x_1(0) = 0,\; x_2(0) = +1$ to define the curve:

$$\Gamma_- = \Big\{ x_1 = -2e^t + 2t + 2, \;\; x_2 = 2e^t - 1, \;\; for \;\; t \geq 0 \Big\}$$

- Observe that all hyperplanes through $(x_{10}, x_{20}) = (0, 1)$ normal to $\eta_0 = (\eta_{10}, \eta_{20}) = (cos\theta, sin\theta)$ for each fixed θ on $\pi \leq \theta \leq 2\pi$ is a supporting hyperplane for the convex set G. The extremal with data $x_{10} = 0$ and $x_{20} = 1$, follows Γ_- as long as $\eta_1(t) + \eta_2(t) < 0$. But

$$\eta_1(t) + \eta_2(t) = (sin\theta - cos\theta)e^{-t} + 2cos\theta$$

Thus for each θ on $\pi \leq \theta \leq 3\pi/2$, we find

$$u(t) = sgn[\eta_1(t) + \eta_2(t)] = -1$$

For each θ on $3\pi/2 \leq \theta \leq 7\pi/4$, there exists a $t(\theta)$ positive zero for $\eta_1(t) + \eta_2(t)$. It is easy to show that (θ) decreases monotonically from $+\infty$ to 0 as θ increases. Thus there exists extremal controllers satisfying the transversality condition $\eta(t_1)x(t_1) = 0$ at G, which switches from $u = 1$ to $u = -1$ at an arbitrarily prescribed point of Γ_-, and then follows for Γ_-.

Define Γ_+ reflecting Γ_- trough the origin to obtain the complete switching locus: $W = \Gamma_+ \cap \Gamma_-$. Note that the corresponding curve $x_2 = W(x_1)$ separates $R^2 - G$.

Define the synthesizing function:

$$\Phi(x_1, x_2) = \begin{cases} -1 & for \;\; x_2 > W(x_1) \;\; and\, on \;\; x_2 = \Gamma_-(x_1) \\ +1 & for \;\; x_2 < W(x_1) \;\; and\, on \;\; x_2 = \Gamma_+(x_1) \end{cases}$$

7 Optimal control for linear process with integral cost criteria.

We will consider now the Maximun Principle for the case where the control process is linear $\dot{x} = A(t)x(t) + B(t)u(t)$ with an integral convex cost.

$$C(u) = C(u(t)) = \int_0^{t_1} f(x(t), u(t))dt, \;\; x(t) = x(t, x_0, u(t)).$$

This principle can be applied to a much wider class of optimal control problems, as we will show there is no restriction to linear state equations.

A system in unstable equilibrium is one in which any deviation from equilibrium increases if the system is left to itself. The application of a control aims to reduce the deviation to zero, but at some cost depending on the size of the control that has to be employed. When there are several successful controls, the choice of one over the other, may be dictated by a *cost* or *performance criterion.* Our problem will then become an *Optimal Control Problem.*

The O.C.P. is to find a controller such that, x_0 will be steered to an state in the target, using a control $u(\cdot)$ from the appropriate class Δ of the admissible controllers in such way that $C(u)$ is a minimun.

For each control $u(t) \in \Omega$ there is a corresponding $x(t)$ with $x(0)$ at some fixed initial state x_0 that satisfies the state equation

$$\dot{x}(t) = f(t, x(t), u(t))$$

Definition 36 *A control u is said to be an* **admissible control** *if there exists a trajectory x corresponding to u such that:*

- *(i) $f^0(t, x(t), u(t))$ is integrable in* $[t_0, t_1]$
- *(ii)* $u(t) \in \Omega$

- *(iii)* $x(t_0) = x_0$ *and* $x(t_1) \in \mathcal{T}$.

The control $u^* \in \Omega$ if optimal if it is *successful,* i.e., if it is appropriate for the problem, and there exists $t_1 \geq 0$ such that $x(t_1, x_0, u(\cdot))$ reaches the target, and is *minimal* that is:

$$C(u^*) \leq C(u) \;\; for\ all\ u \in \Omega.$$

The first problem which we face is the one of *existence of the optimal control.* If the answer to this problem is affirmative then, we face the following one: *How can this optimal control be found?* We will see that the maximal principle is an answer for this question, but in most cases it is not possible to find an analytical solution, computational methods are necessary, but this is not easy matter. We shall not pursue these matters here.

In the differential calculus the minimun of a function of a real variable is located by examining the critical points, those points at which the derivative is zero. In the theory of optimal control we look for those points such that maximize the hamiltonian function:

$$H(x(t), u(t), t) = \eta_0 f(x(t), u(t), t) + \eta(t)[A(t)x + B(t)u(t)]$$

which are called the maximal controllers.

Where $\eta_0 \leq 0$ and $\eta(t)$ is the solution of the adjoint system $\dot{\eta} = -\frac{\partial H}{\partial x}$.

Then we use the **maximal principle:** This principle establishes that: *If a control is optimal then it is a maximal control.* (Note that $u(t)$ is called a maximal controller even thought it yield a minimun cost).

We start considering a criterium for a convex cost with a linear control process more general problems will be considered later.

7.1 Existence of the optimal control for linear processes with integral convex cost.

We now treat the linear control process in R^n :

$$\dot{x} = A(t)x + B(t)u \tag{11}$$

with the integral cost functional:

$$C(u) = g(x(T)) + \int_{t_0}^{T} f_0(t, x) + h_0(t, u)dt \tag{12}$$

where $A(t), B(t), g(x), f_0(t, x)$ and $h_0(t, u)$ are continuous in $[t_0, T]$ and $x \in R^n$ and $u \in R^m$.

Remark 37 *We also assume here that for each fixed t*

- *(a)$f_0(t,x)$ and $h_0(t,u)$ are convex functions,*
- *(b)$f_0(t,x) \geq 0$ and $h_0(t,u) \geq a|u|^p$ for some $a > 0$ and $p > 1$.*

  Hence, $C_0(u) = \int_{t_0}^{T} f_0(t,x) + h_0(t,u)dt \geq a \int_{t_0}^{T} |u(t|^p dt.$

As we will see in theorem (41), the conditions (a) and (b) stated above, assure the existence of (minimal cost) optimal controller among the class of *all measurable controllers with finite cost.* If we assume that the each controller $u(t)$ on $t_0 \leq t \leq T$ lies in a given compact convex restraint set Ω, then we can eliminate any need for positivity or growth bounds on functions f_0 and h_0, as we will see in theorem (42).

If the class of admissible controllers and its associate responses is non empty, it does not necessarily follow that an optimal control exists. The following example illustrate the non-existence of optimal controllers.

Example 38 Non existence of optimal controllers

$$\begin{array}{l} C(u) = \int_0^1 t^2 u^2(t)dt \\ \dot{x} = u(t) \\ x(t_0) = 1 \quad x(1) = 0 \end{array}$$

Let Ω the set of integrable functions.

The trajectory for each $u \in \Omega$ is $x(t) = 1 + \int_0^t u(s)ds$.

For each $0 < \epsilon < 1$ define a control u_ϵ as follows:

$$u_\epsilon = \begin{cases} 0 & if \epsilon \leq t \leq 1 \\ -\epsilon^{-1} & if\ 0 \leq t \leq \epsilon \end{cases}$$

Let x_ϵ the unique trajectory corresponding to u_ϵ satisfying $x(0) = 1$.

$$C(u) = \int_0^\epsilon t^2 \epsilon^{-2} dt = \epsilon/3.$$

So, $\lim_{\epsilon \to 0} C(u = 0$. It is clear that $C(u) = 0$ if and only i $u = 0$ but this is not admissible, because $\phi(0) = 1$. So there does not exist optimal control.

The following example shows that there may be more than one optimal controller.

Example 39 *Let $C(u) = \int_0^1 (1 - u^2(t))dt$ be the cost functional. Let the state equation be: $\dot{x} = u(t)$ and $x(0) = 0, \;\; x(1) = 0$ and $\Omega = \{u : |u(t)| \leq 1.\}$.*

Exercise 7 *Show that for each integer $n = 1, 2, ...,$ the controller $u^*(t) = (-1)^k$ if $\frac{k}{2^n} \leq t \leq \frac{k+1}{2^n}$. $k = 0, 1, 2, ..., 2^n - 1$ are admissible controllers, and $C(u^*) \leq C(u)$ for all admissible controller.*

.

- For notational convenience we define:

$$x_u^0(t) = \int_{t_0}^{t} f_0(t, x) + h_0(t, u) dt.$$

$\bar{K} = \bar{K}(T; x_0) \subset R^{n-1}$ is the set of all response endpoints,

$$\bar{x}_u(T) = (x_u^0(T), x_u(T)) \in R^{n+1},$$

for all admissible control vectors $u(t)$ on $t_0 \leq t \leq T$.

The response $\bar{x}(T)$ can be computed from:

$$x(t) = \Phi(t)x_0 + \Phi(T) \int_{t_0}^{T} \Phi(s)^{-1} B(s)u(s)ds \;\; and, \;\; C(u) = g(x(T)) + x^0(T)$$

Theorem 40 *Consider the control optimal program given by (11) and (12) then: The set of attainability $\bar{K} \subseteq R^{n+1}$ is closed and convex.*

Proof: Let $\bar{x}_1 = (x_1^0, x_1)$ and $\bar{x}_2 = (x_2^0, x_2)$ be two points in $\bar{K}$, corresponding to the controllers $u_1(t)$ and $u_2(t)$ on $0 \leq t \leq T$. Let

$$\bar{y} = (y^0, y) = \lambda \bar{x}_1 + (1 - \lambda)\bar{x}_2, \; 0 \leq \lambda \leq 1.$$

To prove that $\bar{K}$ is convex we must construct a controller steering $(0, x_0)$ to $\bar{y}$.

Define

$$\tilde{u}(t) = \tilde{u}_1(s) + (1 - \lambda)\tilde{u}_2(s).$$

with the corresponding response:

$$x_{\tilde{u}}(s) = \lambda \bar{x}_1(s) + (1 - \lambda)\bar{x}_2(s).$$

So, $x_{\tilde{u}}(T) = y$.

Now from the convexity of $h(t, \cdot)$ it follows that:

$$x_{\tilde{u}}^0(T) \leq \lambda \bar{x}_1^0(T) + (1 - \lambda)\bar{x}_2^0(T) = y^0.$$

But if $(x^0_{\bar{u}}(T), x_{\bar{u}}(T)) \in \bar{K}$ and if $y^0 \geq x^0_{\bar{u}}(T)$ then $(y_0, x_{\bar{u}}(T)) \in \bar{K}$ [4].

The proof that $\bar{K}$ is a closed set is given in [Lee, E.; Markus, L.] 209.

Theorem 41 *If either $g(x)$ is (a) bounded below or (b) is a convex function, then there exists a (minimal cost) optimal controller.*

Proof: Since each allowable control $u(t)$ defines an end point $(x^0(T), x(T)) \in \bar{K}$ we need only prove that the function $g(x) + x^0$ assumes its minimun in $\bar{K}$.

(a) If the hypothesis in the remark (37) are satisfied and $g(x) > a$ then $\lim_{x^0 \to \infty}[g(x) + x^0] = \infty$. Thus, there exists α such that the minimun of $[g(x) + x^0]$ on $\bar{K}$ is assumed on the compact set $\bar{K} \cap [x_0 \leq \alpha]$.

(b) Assume that $g(x)$ is a convex function not bounded below. Then, for each real number c_1 the set in R^{n+1} defined by $C = \{x : g(x) + x^0 \leq c_1\}$ is closed, has a non empty interior (because $g(x)$ is not bounded below) and it is also convex.

(b1) To prove the convexity of the set C consider:

$$g(x_1) + x^0_1 \leq c_1 \ \ \textit{and} \ \ g(x_2) + x^0_2 \leq c_1,$$

it follows that

$$g[\lambda x_1 + (1 - \lambda)x_2] + \lambda x^0_1 + (1 - \lambda)x^0_2 \leq c_1.$$

Now consider a constant c_1 such that the set $g(x) + x^0 \leq c_1$ meet K. This intersection is closed and convex. As we shall see, this intersection is also bounded and hence compact. Then the existence of the optimal control follows.

(b2.) To prove the boundedness of this intersection:

- Consider the hyperplane π as the supporting hyperplane in R^{n+1} to the convex set C such that for (x^0, x) below π.

[4] To see this, construct $\bar{u}(t) = \bar{u}(t) + u_\beta(t)$ such that:

$$\int_{t_0}^{T} \Phi(s)^{-1} B(s) u_\beta(s) ds = 0$$

and

$$x^0_u(T) + b = \int_{t_0}^{T} f_0(t, x) + h_0(t, \bar{u}) dt.$$

- We shall show that for points $(x^0, x) \in \bar{K}$ such that $|x(T)|$ is large enough then $x^0(T) \geq k|x(T)|$ for a prescribed constant $k > 0$. Such point must lie above π and hence satisfy $g(x) + x^0 > c$. Then $C \cap \bar{K}$ is bounded.

Upon this calculation, we obtain the require compactness then, the existence of the minimun is proved.

We now prove the boundedness of $C \cap \bar{K}$.

For points $(x^0, x) \in \bar{K}$ we have:

$$|x(T)| \leq |\Phi(T)x_0| + \int_{t_0}^{T} |\Phi(T)\Phi(s)^{-1}B(s)||u(s)|ds$$

Using the assumptions on $f^0(t,x)$ and $h^0(t,u)$ if $|x(T)| \geq 2|\Phi(T)x_0|$ and we write: $M \geq |\Phi(T)\Phi(s)^{-1}B(s)|$ then: $\int_{t_0}^{T} |u(s)|ds \geq |x(T)|/2M$. By Schwarz' inequality: $\int_{t_0}^{T} |u(s)|ds \leq c_2 \left[\int_{t_0}^{T} |u(s)|^2 ds\right]^{\frac{1}{2}}$ for a constant c_2.

Then we obtain: $|x(T)|^p \leq 4M^p c_2 \int_{t_0}^{T} |u(s)|^p ds \leq c_3 x^0(T)$. Hence for sufficiently large $|x(T)|$ we have: $x^0(T) \geq |x(T)|$, and $(x^0(T), x(T)) \in K$ lies above the hyperplane π.

Therefore the closed intersection of $\{x : g(x) + x^0 \leq c_1\}$ and K is bounded, hence compact.

If we consider the linear control process in R^n : given by (11) and (12), where the matrices of coefficient are continuous and $g(x)$, $f_0 = (x,t)$ and $h_0(x,t)$ are continuous for all values of this argument, if in addition of the $f_0 = (x,t)$ and $h_0(x,t)$ are convex in each $t_0 \leq t \leq T$ we assume that each controller $u(t)$ on $t_0 \leq t \leq T$ lies in a given compact convex restraint set Ω, then we can eliminate any restriction of positivity or grounds bounds on the functions $f^0(x,t)$ and $h^0(x,t)$.

Theorem 42 *Consider the control process in R^n*

$$\dot{x} = A(t)x + B(t)u$$

with cost functional

$$C(u) = g(x(T)) + \int_{t_0}^{T} f_0(x,t) + h_0(t,u)dt$$

and compact convex restraint $\Omega \subset R^m$. Then there exists an optimal controller.

Proof: Wee seek the minimun of the real continuous function $g(x) + x^0$ on the bounded $\bar{K} \subset R^{n+1}$. s this function decreases monotonically with x_0 for each x fixed, the infimun of $g(x) + x^0$ is just the minimun, on the lower boundary of $\bar{K}$. The required minimun is assumed, see [Lee, E.; Markus, L.].

7.2 Necessary and sufficient conditions for optimality in the case of linear processes

We shall prove in this section that an optimal control is necessarily extremal. We begin the section characterizing the condition of extremal controllers, and then we obtain necessary and sufficient conditions for extremality. Finally we obtain necessary and sufficient conditions for optimality.

Definition 43 *Given a control problem in R^n, with set of attainability $\bar{K} \subseteq R^{n+1}$ corresponding to the cost functional $C(u)$, a control $\bar{u}(t)$ on $t_0 \leq t \leq T$, which steers $(0, x_0)$ to a relative boundary point of $\bar{K}$ is called an* **extremal control** *and the corresponding response $\bar{x}(t)$ is also extremal.*

7.3 The maximal principle for linear processes with convex cost functional.

Theorem 44 *Consider the control process in R^n given by (11) and with the integral cost functional:*

$$C(u) = \int_{t_0}^{T} f_0(t, x) + h_0(t, u)dt \tag{13}$$

A controller $\bar{u}(t)$ with response $\bar{x}(t)$ is extremal if and only if there exists a vector $\bar{\eta}(t) = (\eta_0, \eta(t))$ satisfying:

$$\begin{array}{l} \dot{\eta}_0 = 0, \ \eta_0 < 0 \\ \\ \dot{\eta} = \eta_0 \frac{\partial f_0}{\partial x}(t, \bar{x}(t)) - \eta A(t) \end{array} \tag{14}$$

and such that the maximal principle holds almost everywhere:

$$\eta_0 h_0(t, u^*) + \eta(t)B(t)u^* = \max_u [\eta_0 h_0(t, u) + \eta(t)B(t)u].$$

Proof: Let $\bar{u}(t)$ be a controller with response $\bar{\bar{x}}(t) = (\bar{x}^0(t), \bar{x}(t))$ and adjoint response $\bar{\bar{\eta}}(t) = (\eta_0, \eta(t))$ satisfy: (11) , and $\dot{x}^0 = f_0(t, x) + h_0(t, u)$, $\bar{\bar{x}}(t_0) = (0, x_0)$, and also the system (14) and the above maximal principle. We shall prove that

$$\bar{\bar{\eta}}(T)\bar{\bar{x}}(T) \geq \bar{\bar{\eta}}(T)\bar{\bar{\omega}}(T),$$

where $\bar{\bar{\omega}}(T) = (\omega^0(t), \omega(t))$ is the response to an arbitrary admissible controller $u(t)$.

From this inequality it follows that $\bar{\bar{x}}(T)$ lies on the boundary of $\bar{K}$ and that $\bar{\bar{\eta}}(T)$ is an exterior normal to $\bar{K}$.

Use the equality:

$$\frac{d}{dt}[\bar{\bar{\eta}}(t)\bar{\bar{\omega}}(t)] = \eta_0 \dot{w}^0 + \dot{\eta} w + \eta \dot{w},$$

and compute: $\bar{\bar{\eta}}(T)\bar{\bar{\omega}}(T) - \bar{\bar{\eta}}(t_0)\bar{\bar{x}}_0$

$$= \int_{t_0}^{T} \eta_0 \left[f_0(t,w) - \frac{\partial f_0}{\partial x}(t,\bar{x})w \right] + [\eta_0 h_0(t,u) + \eta B u] dt.$$

Analogously for: $\bar{\bar{\eta}}(T)\bar{\bar{x}}(T) - \bar{\bar{\eta}}(t_0)\bar{\bar{x}}_0$,

$$= \int_{t_0}^{T} \eta_0 \left[f_0(t,\bar{x}) - \frac{\partial f_0}{\partial x}(t,\bar{x})\bar{x} \right] + [\eta_0 h_0(t,\bar{u}) + \eta B \bar{u}] dt.$$

Using now the maximal principle and the convexity condition

$$f_0(t,w) - f_0(t,\bar{x}) \geq \frac{\partial f_0}{\partial x}(t,\bar{x})(w - \bar{x}),$$

then the claim follows.

Conversely, assume that $\bar{u}(t)$ is extremal, so that the corresponding response $\bar{\bar{x}}(t) = (\bar{x}^0(t), \bar{x}(t))$ steers $(0, x_0)$ to $\bar{\bar{x}}(t)$ on the boundary of $\bar{K}$. Let $\bar{\bar{\eta}}(T) = (\eta_0, \eta(T))$ be an exterior normal to $\bar{K}$ at $\bar{x}(T)$. Clearly $\eta_0 < 0$ then, we can assume that $\eta_0 = -1$. Let $\bar{\bar{\eta}}(t)$ be defined as the solution of the adjoint system $\mathcal{A}$ with the given data $\bar{\bar{\eta}}(T)$. We must prove that

$$-h_0(t,u^*) + \eta(t)B(t)u^* = \max_u[-h_0(t,u) + \eta(t)B(t)u],$$

almost everywhere on $t_0 \leq t \leq T$.

Essentially, this method consists in adding an impulse perturbation, to $\bar{u}(t)$ over a short duration $t_1 \leq t \leq t_1 + \epsilon$, where we suppose that $\bar{u}(t)$ fails to satisfy the maximal principle.
The perturbed controller $u^*(t)$ yields an increment in the terms $\int_{t_0}^{T}[\eta_0 h^0(t,u) + \eta(t)B(t)u]dt$ in the computation of $\bar{\bar{\eta}}(Y)\bar{w}(T)$, which contradicts the assertion that $\bar{x}(T)$ lies on the boundary of $\bar{K}$.

The details of the proof are in [Lee, E.; Markus, L.] 213.

Remark 45 A comment on uniqueness. *In the case where $g(x)$ is convex and $h(t,u)$ is strictly convex, any two extremal controllers steering $(0, x_0)$ to the same boundary point of $\bar{K}$ must coincide almost everywhere*

Theorem 46 *Consider the control process in R^n given by (11) and (12). Assume that $g(x) \in C^1$ is convex in R^n. Then there exists a solution $x^*(t), \eta^*(t)$ of the system:*

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u^*(t,\eta) \\ \dot{\eta} &= \tfrac{\partial f_0}{\partial x}(t,x(t)) - \eta A(t) \end{aligned} \tag{15}$$

with $x_(t_0) = x_0,\ \eta(T) = -grad\ g(x(T))$.

where $u^*(t,\eta)$ is defined by the maximal principle

$$-h_0(t,u^*) + \eta(t)B(t)u^* = \max_u[-h_0(t,u) + \eta B(t)u], \tag{16}$$

An optimal controller *is* $u^*(t) = u^*(t,\eta^*(t))$ *with the corresponding optimal response* $x^*(t)$.

- If $h_0(t,u)$ is strictly convex for each t, then the solution $x^*(t), \eta^*(t)$ is unique and $u^*(t)$ is the unique optimal controller with the optimal response $x^*(t)$. Therefore, in this case the principle of the maximum is also a sufficient condition for the optimality of the control and its associate response.
- If $h_0(t,u)$ is merely convex we choose $u^*(t,\eta)$ selecting the point in R^m which has minimal coordinates among all solutions of the maximal principle. That is choose $u^*(t,\eta) = (u_1^*, ..., u_m^*)$ so that u_1^* is minimal among all possible solutions of the MP., then choose u_2^* as minimal among all solutions with designed value of u_1^*, continue in this manner. The $u^*(t) = u^*(t,\eta)$ is an admissible controller.

We shall give the proof of the theorem, following several steps:

1. We first show that there exists a unique constant m such that the set

$$S_m = \left\{ \bar{x} \in R^{n+1} : g(x) + x^0 \leq m \right\}$$

   is tangent to $\bar{K}$ but is separated from the relative interior of $\bar{K}$ by a common supporting hyperplane π^*.

2. We show that m is the optimal cost.

3. We find a solution $x^*(t), \eta^*(t)$ for the nonlinear boundary-value problem.

4. If $h^0(t,u)$ is strictly convex for each fixed t, then the optimal control $u^*(t)$ and its response $x^*(t)$ are unique.

5. Then $(x^*(t), \eta^*(t))$ are unique.

   Proof of the step 1. As we know, the intersection of $\bar{K}$ with S_c is compact for large c. Hence we define m as the infimun of all such numbers c such that the corresponding intersection is not empty. For $c > m$ the hypersurface S_c meets the relative interior of $\bar{K}$, and for $c < m,\ S_c$ does not meet $\bar{K}$. Thus only for $c = m$ can S_c be tangent to $\bar{K}$.

Proof of the step 2. It follows immediately from step 1.

Proof of the step 3. Let $\bar{\eta}^*(T) = (-1, \eta^*(T))$ be the normal to the tangent hyperplane to S_m at some point $P \in S_m \cap \bar{K}$. Let $u^*(t)$ be an extremal control, steering $(0, x_0)$ to $P = \bar{x}^*(T)$ by the response $\bar{x}^*(t) = (x^{0*}(t), x^*(t))$. Let $\bar{\eta}^*(t) = (-1, \eta^*(t))$ be defined as the solution of

$$\dot{\eta} = \frac{\partial f^0}{\partial x}(t, x^*(t)) - \eta A(t),$$

with $\eta^*(T) = -grad\ g(x^*(T))$.

From the previous theorem, we find that $u^*(t)$ satisfies the maximal principle with the adjoint response $\bar{\eta}^*(t)$ thus $x^*(t), \eta^*(t)$ is the solution of the nonlinear boundary-value problem.

Proof of the step 4. If $h^0(t, u)$ is strictly convex for each fixed t, then $S_m \cap \bar{K}$ is just a single point P. In other case, let P_1 and P_2 two points in $\bar{K} \cap S_m$, and thus in the relative boundary of $\bar{K}$.

Let $u_1(t)$ and $u_2(t)$ be extremal controls with responses $x_1(t)$ and $x_2(t)$ leading to $P_1 = (x_1^0(T), x_1(T))$ and $P_2 = (x_2^0(T), x_2(T))$ respectively.

Consider the control $u' = \frac{1}{2}[u_1(t) + u_2(t)]$ with response $\bar{x}'(t) = (x'^0(t), x'(t)$ then

$$x'(T) = \frac{1}{2}[x_1(t) + x_2(t)]$$

and from the strict convexity of $h(t, u)$ for each t we obtain that

$$x'^0(T) < \frac{1}{2}[x_1^0(t) + x_2^0(t)].$$

The half line $x^0 > x^0(T), x = x(T)$ lies in the relative interior of $\bar{K}$, which implies that the midpoint between P_1 and P_2 lies in the relative interior of $\bar{K}$. This contradiction show that $S_m \cap \bar{K}$ consists of a single point P.

Proof of the step 5. From step 4 it follows that $x^0 + g(x)$ can assume its minimun at just one single point $P \in \bar{K}$. Then, from the strict convexity of $h^0(t, u)$ for each t, it follows that there is one single optimal control $u^*(t)$ which steers $(0, x_0)$ to the point P. In other case, consider the controller $u' = \frac{1}{2}[u_1(t) + u_2(t)]$ and then from the convexity of h^0 it follows that $h^0(t, u') + \eta(t)B(t)u'(t) > m(t)$, whenever $u_1(t) \neq u_2(t)$.

So the optimal control and its response $x^*(t)$ are unique. Also $\eta^*(t)$ is uniquely determined as the solution of a linear differential system with the data: $\eta(T) = -grad\ g(x^*(T))$.

7.4 Some remarks on restricted end point problems.

- Suppose that x_1 is the prescribed target, and suppose that the initial state x_0 is controlled with minimal cost to x_1. We also assume that $h_0(t,u)$ is strictly convex for each t. Under this condition there exists a unique optimal controller $u*(t)$ In fact, let l be the line $x = x_1$, in R^{n+1} then $l \cap \bar{K}$ is a closed horizontal ray. The control $u^*(t)$ steers $(0, x_0)$ to $(x^0(T), x_1)$ where $x^0(T)$ is the lowest point in the ray $l \cap \bar{K}$.

- If the target $\mathcal{T}$ in R^n is defined by a function γ, convex and C^1

$$G = \{x : \gamma(x) \leq 0\}$$

such that $grad\ \gamma \neq 0$ on $fr(G)$, the boundary conditions are: $x(0) = x_0,\ \gamma(x(T)) = 0$ recall that if u is extremal then the associate response x must satisfy $x(T) \in fr(G)$, and $\eta(T) = -k\ grad\ \gamma(x(T))$, for some $k > 0$.

- In the case where $g(x) \equiv 0$ then $C(u) = C_0(u)$ then the minimal value $x^{0*} \in \bar{G} \cap \bar{K}$ occurs at just one common boundary point $\bar{x}^*(T) = (x^{0*}(T), x^*(T))$.

- In the special case where the optimal controller minimizing $C_0(u)$ makes no reference to a target, that is $x(T)$ is free, then the corresponding value is $\eta(T) = 0$.

7.5 Examples

Example 47 *A simple controlled dynamical system is modelled by the scalar equation*

$$\dot{x} = x + u,$$

and we employ the cost functional:

$$C(u) = \frac{1}{4}\int_0^1 u(t)^4 dt.$$

The control problem consists in steering $x(t)$ from an initial state $x(0) = x_0$ to the target $x(1) = 0$, with minimal cost.

The maximal principle states (16):

$$-\frac{u^{*4}}{4} + \eta u^* = \max_u \left[-\frac{u^4}{4} + \eta u\right]$$

or:

$$u^* = (\eta)^{\frac{1}{3}}.$$

Thus we must solve:

$$\dot{x} = x + \eta^{\frac{1}{3}}, \quad \dot{\eta} = -\eta.$$

with $x(0) = x_0,\ x(1) = 0$. Since $\eta = \eta_0 e^{-t}$ we have:

$$x = e^t x_0 - \frac{3}{4}\eta_0^{\frac{1}{3}}[e^{-\frac{t}{3}} - e^t].$$

The boundary conditions yield:

$$\eta_0^{\frac{1}{3}} = \frac{4x_0}{3}(e^{-\frac{4}{3}} - 1)^{-1}$$

and the optimal controller is

$$u^* = \frac{4x_0}{3}(e^{-\frac{4}{3}} - 1)^{-1}e^{-\frac{t}{3}}$$

Example 48 *Consider the Eisener-Stroz model focused on net investment as a process that expands a firm's plant size. Assuming that the firm has knowledge of the profit rate π associated which each plant size, as measured by the capital stock K. We have a profit function $\pi(K)$. To expand the plant and adjustment cost C is incurred whose magnitude varies positively with the speed of expansion $\bar{K}(t)$. Then he have a increasing function $C(K')$. The derivative $\bar{K}(t)$ is the net investment, $I = \bar{K}(t)$.*

The objective of the firm is to choose an optimal path $K(t)$ that maximizes the total present value of its net profit over time:

$$\begin{aligned} &\textit{Maximize} \quad \textstyle\int_0^T [\pi(K) - C(I)]e^{-\rho t}dt \\ &\textit{subject to} \quad \dot{K} = I \\ &\textit{and} \qquad\quad K(0) = K_0, \ \ K(T) = K_T. \end{aligned}$$

We assume that both the π and C function are quadratic:

$$\begin{aligned} \pi &= \alpha K - \beta K^2 \quad (\alpha, \beta > 0) \\ C &= aI^2 + bI \qquad (a, b > 0) \end{aligned} \tag{17}$$

The maximal principle states:

$$-C'(I^*)e^{-\rho t} + \eta = \max_{0 \le I \le 1}\left\{-C'(I)e^{-\rho t} + \eta\right\}$$

Taking derivatives and using(17), we obtain that:

$$I^* = \eta\frac{e^{\rho t} - b}{2a}$$

Thus we must solve:

$$\dot{K} = I^* \;\; and \;\; \dot{\eta} = \pi'(K)e^{-\rho t}$$

Taking account of equation (17), it follows that: $\eta = \frac{(\alpha+2\beta K)}{\rho}e^{-\rho t} + k_0$

So the optimal controller is:

$$I^*(t) = \frac{(\alpha + 2\beta K)}{\rho 2a} - \frac{b}{2a} + \frac{k_0 e^{\rho t}}{2a}.$$

Substituting in the state equation it follows:

$$\dot{K} - \frac{\beta}{a\rho}K = \left[\frac{\alpha}{2a\rho} - \frac{b}{2a}\right] + \frac{k_0}{2a}e^{\rho t}.$$

The optimal path has a qualitative form given by:

$$K^*(t) = Ae^{\rho_1 t} + \bar{A}$$

Where $\bar{A} = \frac{a\rho}{\beta}\left[\frac{\alpha}{2a\rho} - \frac{b}{2a}\right]$ and A can be found from the initial conditions.

7.6 Regulation over an infinite interval

We next allow the time interval $t_0 \leq t \leq T$ to become infinite.

Theorem 49 *Consider the controllable autonomous process in R^n*

$$\dot{x} = A(t)x + B(t)u \tag{18}$$

with the integral cost functional:

$$C(u) = \int_{t_0}^{\infty} f^0(x) + h^0(u)dt \tag{19}$$

where $f^0(x) \geq 0$ is convex, $f(x) = 0$ if and only if $x = 0$, $h_0(u) \geq a|u|^p$ is strictly convex and $h_0(0) = 0$. Then there exists an unique optimal controllers $u^*(t)$ on $0 \leq t < \infty$ with response $x^*(t)$.

Assume that no eigenvalue of A has zero real part. Then a necessary and sufficient condition *that an admissible controller $\bar{u}(t)$ with response $\bar{x}(t)$ on $0 \leq t < \infty$ be optimal is that the maximal principle obtains:*

$$\bar{\eta}_0 h_0(\bar{u}(t)) + \bar{\eta}(t)B\bar{u}(t) = \max_u[\bar{\eta}_0 h_0(u(t)) + \bar{\eta}(t)Bu(t)]$$

where $\bar{\eta}(t) = (\bar{\eta}_0, \bar{\eta}(t))$ satisfies the adjoint system:

$$\begin{array}{l} \dot{\eta}_0 = 0 \\ \dot{\eta} = -\eta_0 \frac{\partial f_0}{\partial x}(\bar{x}(t)) - \eta A \end{array} \tag{20}$$

with $\bar{\eta}_0 < 0$ and $\bar{\eta}(\infty) = 0$.

Proof: Since (18) is controllable. the initial state x_0 can be steered to the origin at $t = 1$, and then kept fixed at the origin by the null control $u = 0$. In this way there exists an admissible controller having finite cost M. Next we shall construct the optimal controller $u^*(T)$ on $0 \leq t < \infty$ of the weak limit of the appropriate optimal controllers on finite time intervals.

Using the given initial state x_0 at $t = 0$ to obtain the optimal controller $u^*(t)$ on each finite interval $0 \leq t \leq k$, for $k = 1, 2, ...$ with the cost functional $C_k(u) = \int_{t_0}^{k} f^0(x) + h^0(u)dt$, write this minimal cost $C_k(u{*}_k) = m_k$ and note $m_k \leq m_{k+1} \leq M$, since $u^*_{k+1}(t)$ cannot have a smaller cost than $u^*_k(t)$. Since $\int_0^\infty |u^*_k(t|)^p dt \leq M/a$ (we can define $u^*_k(t) \equiv 0$, for $t > k$), we can select a subsequence $u^*_{k_i}(t)$ which converges weakly to a limit $u^*(t)$ on each compact time interval [5] For each finite $T > 0$,

$$\int_{t_0}^{T} f^0(x^*) + h^0(u^*)dt \leq \lim_{k_i \to \infty} inf \int_{t_0}^{T} f^0(x^*_{k_i}) + h^0(u^*_{k_i})dt \leq \lim_{k_i \to \infty} \leq M.$$

Therefore $u^*(t)$ is an admissible controller with finite cost $C(u^*) = m \leq M$.

To show that $m = \lim_{k \to \infty} m_k$ and $u^*(t)$ is the unique optimal controller on $0 \leq t < \infty$. see [Lee, E.; Markus, L.] 221.

We now show that $\eta^{*0} < 0$ and $\eta^*(\infty) = 0$. If $\eta^{*0} = 0$ then $\eta^*(t)$ would be zero. Since $\lim \eta_{k_i \to \infty}(t) = \eta^*(t)$ and $\eta_{k_i}(k_i) = 0$. Using the uniform convergence on compact of $(\partial f^0/\partial x)(x^*_k)$ to $(\partial f^0/\partial x)(x^*_k)$, it follows that

$$\eta^*(t) = \eta^*(0)e^{-At} + \int_0^t -\eta^0 \frac{\partial f^0}{\partial x}(x^*(s))e^{-A(t-s)}ds.$$

If every eigenvalue of A has positive real part, then $|e^{-At}| < ce[-\lambda t$ on $0 \leq t < \infty$ for a constant c. Using $\frac{\partial f^0}{\partial x}(x^*((t)) \to 0$, it is easy to prove that $\eta^*(\infty) = 0$ and we omit the details.

After a linear change of variables on η we can suppose that

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}$$

where every eigenvalue of A_+ has positive real part (and hence the corresponding coordinates of $\eta(\infty)$, $\eta^*_+(\infty) = 0$), and every eigenvalue of A_- has negative real part. Now it is sufficient to prove that the corresponding coordinates $\eta^*_+(\infty) = 0$. See [Lee, E.; Markus, L.] 225.

[5] A sequence $\{u_n(t)\}$ $n = 1, 2, 3, ...$ of real (or vector valued) integrable functions on a real interval $\mathcal{J}$ is called *weakly convergent* to $u^*(t)$ in case, for each bounded measurable test function $g(t)$, we have

$$\lim n \to \infty \int_{\mathcal{J}} g(t)u_n(t)dt = \int_{\mathcal{J}} g(t)u^*(t)dt.$$

The collection of all vector functions $u(t)$ which are measurable on a given finite interval $\mathcal{J}$ and have values in a given compact convex set $\Omega \subset R^n$ is know to be (sequentially) weakly compact. That is, each such sequence of functions has a subsequence that converges weakly on $\mathcal{J}$ to a function of the given collection.

To prove that and admissible controller $\bar{u}(t)$ with response $\bar{x}(t)$ and $(\bar{\eta}_0, \bar{\eta}(t))$ on $0 \leq t < \infty$, such that the maximal principle holds and $\bar{\eta}^0 < 0, \bar{\eta}(\infty) = 0$ is the unique optimal controller, note that $x(\infty) = w(\infty)$ and use step 5 of the previous theorem to show that:

$$\bar{\eta}^0 \bar{x}^0(T) + \bar{\eta}(T)\bar{x}(T) \geq \bar{\eta}^0 w^0(T) + \bar{\eta}(T)w(T)$$

for each finite $T > 0$. Since each of these terms has a limit as $T \to \infty$ and $\eta^0 < 0$ we find

$$C(\bar{u}) \leq C(u).$$

Thus $\bar{u}$ is optimal.

Example 50 Supply of new homes.

Consider a representative firm in the housing market, that has to choose an investment level I in order to maximizes its profits. The profits are given by:

$$\int_0^\infty [P(t)I(t) - C(I(t))]e^{-rt}dt$$

where $C(I)$ denotes the industry cost corresponding to the gross investment I, and $P(t)$ is the competitively determined (stock) price of a standard unit of housing at time t. To simplify our example We shall consider a quadratic cost function, $C(I) = \alpha I^2$. Gross housing investment is the output of the construction industry, defined in the usual way:

$$I = \dot{K} + \delta K$$

assuming a exponential depreciation rate δ.

The optimization program is given by:

$$\max_I \int_0^\infty [P(t)I(t) - \alpha I(t)^2]e^{-rt}dt$$

$$s.a. \quad \dot{K} = I - \delta K.$$

In this case: $f_0(K) \equiv 0$ and $h_0(I(t)) = P(t)I(t) - \alpha I(t)^2$. From the previous theorem we obtain: that the optimal control $\bar{I}$ maximize:

$$\pi(I(t)) = \bar{\eta}_0[P(t)I(t) - \alpha I(t)^2]e^{-rt} + \bar{\eta}I. \tag{21}$$

where $(\bar{\eta}^0, \bar{\eta})$ solves the adjoint system:

$$\dot{\bar{\eta}}^0 = 0$$
$$\dot{\bar{\eta}} = \bar{\eta}\delta.$$

So; $\bar{\eta(t)} = \eta_0 e^{\delta t}$. So; $\bar{\eta(t)} = \eta_0 e^{\delta t}$ and taking derivatives in (21),

$$-[P(t) - 2\alpha I]e^{-rt} + \eta_0 e^{\delta t} = 0$$

then:

$$\bar{I}(t) = \eta_0 e^{(\delta+r)t} + P(t)$$

from the initial condition $I(0) = I_0$ we obtain the value of η_0.

Observe that the transversality condition implies $\eta_0 = 0$. So, $I(t) = \frac{P(t)}{2\alpha}$.

8 Necessary and Sufficient condition for an optimal control with a non linear process: A particular case.

In the previous section the maximal principle was shown to be necessary and sufficient under convexity assumptions, for the optimality of a controller for certain linear processes, we shall see a similar result for particular processes in which the control es effected trough a convex function and where the control process in R^n is the following:

$$\dot{x} = A(t)x + h(u,t) \qquad (\mathcal{S})$$

with initial state $x(t_0) = x_0$ and closed convex target set $G \subseteq R^n$.

The cost functional is

$$C(u) = \int_{t_0}^{T} [f^0(x(t),t) + h^0(u(t),t)]dt$$

or $x^0(T) = C(u)$ where $x^0(t)$ is defined by the scalar differential equation

$$\dot{x}^0 = f^0(x,t) + h^0(u,t), \ \ and \ \ x^0(t_0) = x_0.$$

The admissible controllers $u(t)$ are all bounded measurable $m - vectors$ functions on the fixed interval $[t_0, T]$ steering x_0 to some point in $\mathcal{T}$ and lying in some nonempty restrain set $\Omega \subset R^n$.

Theorem 51 *In the above conditions, let the coefficients $f^0, \partial f^0/\partial x, h^0, A,$ be continuous and $f^0(x,t)$ is convex in x for each t such that $t_0 \leq t \leq T$. Assume that $u^*(t)$ is a controller with response $x^*(t) = (x^{0*}(t), x^*(t))$ satisfying the maximal principle:*

$$-h^0(u^*(t),t) + \eta(t)h)u^*(t),t) = \max_{u\in\Omega}[h^0(u(t),t) + \eta(t)h(u(t),t)$$

for almost all t. Here $\eta(t)$ is any nontrivial solution of

$$\dot{\eta} = \frac{\partial f^0}{\partial x}(x^*(t),t) - \eta A(t).$$

satisfying the transversality condition:

$\eta(T)$ is inward normal of $\mathcal{T}$ at the boundary point $x^*(T)$. (If $G = R^n$ then $\eta(T) = 0$; if $G = x_1$ is a single point, the condition is vacuous.)

Then $u^*(t)$ is an optimal controller achieving the minimal cost:

$$C(u^*) = x^{0*}(T).$$

Proof: Let $u^*(t), \bar{x^*}(t)$, and $\eta^*(t)$ satisfy the maximal principle and transversality conditions, and let $u(t)$ be any admissible controller with response $\bar{x}(t) = (x^0(t), x(t))$ on $t_0 \leq t \leq T$. We shall first prove the basic inequality

$$-x^{*0}(T) + \eta(T)x^*(T) \geq -x^0(T) + \eta(T)x(T).$$

Compute the derivative:

$$\frac{d}{dt}[-x^{*0}(t) + \eta(t)x^*(t)] = -\dot{x}^0(t) + \dot{\eta}(t)x(t) + \eta(t)\dot{x}(t).$$

Use the differential systems for $\dot{x}^0(t)$ and $\dot{x}(t)$ and integrate over the basic interval $t_0 \leq t \leq T$ to obtain:

$$[-x^{*0}(T) + \eta(T)x^*(T)] - \eta_0(t_0)x_0 =$$
$$\int_{t_0}^{T} \left[\frac{\delta f^0(x^*, t)}{\partial x} x - f^0(u, t) - \eta h(u, t) - h^0(u, t) + \eta h(u, t) \right] dt.$$

Next specialize this formula to the control $u^*(t)$ with the response $\bar{x}(t)$ and subtract these inequalities to obtain:

$$[-x^{*0}(T) + \eta(T)x^*(T)] - [x^0(T) + \eta(T)x(T)]$$
$$= \int_{t_0}^{T} \left\{ [-h^0(u^*, t) + \eta h(u^*, t)] - [-h^0(u, t) + \eta h(u, t)] + f^0(x, t) - f^0(x^*, t) + \frac{\delta f^0(x^*, t)}{\partial x}(x^* - x) \right\} dt.$$

But the integrand is almost everywhere positive because of the assumptions of maximal principle for $u^*(t)$ and the convexity of $f^0(x, t)$. Thus the basic inequality is proved.

If $G = R^n$ the transversality condition asserts that $\eta(T) = 0$ and hence $-x^{0*}(T) \geq x^0(T)$, or $C(u) \leq C(u^*(t)$, for every admissible controller $u(t)$. Hence $u^*(t)$ is optimal in this case.

Next let $\mathcal{T}$ be a closed convex set in R^n and let π be a supporting plane to $\mathcal{T}$ at $x^*(T)$ with inward normal $\eta(T)$ which could be zero. Then

$$x^0(T) - x^{0*}(T) \geq \eta(t)(x(T) - x^*(T)).$$

But $x(T)$ lies in G, hence $x(T)$ lies on the inward side of π and $\eta(t)(x(T) - x^*(T)) \geq 0$..Thus $x^{0*}(T) \leq x^0(T)$ and $u^*(t)$ is an optimal controller.

Corollary 52 *In the conditions of the theorem and cost:*

$$C(u) = g(x(T)) + x^0(t)$$

where $g(x)$ is a differentiable convex function. Let $u^*(t)$ satisfying the maximal principle, and the transversality condition:

$$\eta(T) = grad\ g(x^*(T)).$$

Then $u^*(t)$ is an optimal control.

Proof: The basic inequality relating $u^*(t)$ and its response $x^*(t)$ to any other admissible control and response still holds,

$$-x^{*0}(T) + \eta(T)x^*(T) \geq -x^0(T) + \eta(T)x(T).$$

Using the transversality condition, and the convexity of $\mathcal{T}$ we conclude:

$$x^0(T) - x^{*0}(T) \geq -grad\ g(x^*(T))(x(T) - x^*(T)) \leq -[g(x(T) - g(x^*(T))]$$

Therefore:

$$x^0(T) + g(x(T) \leq x^{*0}(T) + g(x^*(T)).$$

9 The maximal principle and the existence of optimal controllers for non linear processes

In this section we show, for the general case of a non linear non autonomous system with moving targets with finite or infinite time durations, necessary conditions for the existence of an optimal control. Next we shall restrict ourselves to the non-autonomous case and we shall see how to find this optimal control.

The set of conditions that characterize an optimal controller is collectively known as the Pontriaguin Maximun Principle (PMP). For many important problems, the conditions of the PMP will only be satisfied by a small subset of our control class (perhaps only by a single control). In this case there is a reasonable chance of our finding an optimal control if one exists.

We shall show that maximal principle together with the transversality conditions, are necessary conditions for an optimal control, see theorem (54). The proof of the maximal principle (that is the necessary condition for an extremal controller)is given further on, see section (14).

9.1 The existence of the optimal control

In this section we shall discuss a theorem that guarantees the existence of optimal control when Ω is compact. The theorem covers an important class of problems that arise in applications.

Theorem 53 *Consider the process in R^n*

$$\dot{x} = f(x, t, u)$$

where f is in C^1 in R^{n+m+1}, let the following hypothesis holds:

1. *The initial and target sets X_0 and $X(t_1)$ are nonempty compact sets varying continuously in R^n for all t in the basic prescribed compact interval $[\tau_0, \tau_1]$*

2. *The control restraint set $\Omega(x, t)$ is a nonempty compact set in R^n for $(x, t) \in R^n \times [\tau_0, \tau_1]$.*

3. *The state constraints are (possibly vacuous) $h^1(x) \leq 0, ..., h^r(x) \leq 0$, a finite or infinite family of constraints, where $h^1, ..., h^r$ are real continuous function on R^n.*

4. *The family $\mathcal{F}$ of admissible controllers consists of all measurable functions $u(t)$ on various intervals $t_0 \leq t \leq t_1$ in $[\tau_0, \tau_1]$ such that each $u(t)$ has response $x(t)$ on $t_0 \leq t \leq t_1$ steering $x(t_0) \in X_0(t_0)$ to $x(t_1) \in X_1(t_1)$ and $u(t) \in \Omega(x, t)$, $h^1(x(t) \leq 0, ..., h^r(x(t)) \leq 0$.*

   The cost functional for each $u \in \Delta$ is

   $$C(u) = g(x_0, t_0, x_1, t_1) - \int_{t_0}^{t_1} f^0(x, t, u)dt + max_{t_0 \leq t \leq t_1} \gamma(x(t)).$$

   Where $f^0 \in C^1(R^{n+m-1})$, and $g(x)$ and $\gamma(x)$ are continuous in R^n.

Assume:

- *(a)The family Δ of admissible controllers is non empty.*
- *(b)There exists a uniform bond $|x(t)| \leq b$ on $t_0 \leq t_1$ for all admissible response $x(t)$*
- *(c) The extremal velocity set $\hat{V}(x, t) = \{f^0(x, t, u), f(x, t, u) : u \in \Omega(x, t)\}$ is convex in R^{n+1} for each fixed (x, t).*

Then there exists an optimal controller $u^*(t)$ on $t_0^* \leq t \leq t_1^*$ in Δ minimizing $C(u)$.

The proof of this theorem is given for instance in [Lee, E.; Markus, L.] or in [Berkovitz, L.D.].

9.2 The Pontriaguin Maximal Principle

Let us now consider an autonomous nonlinear process in R^n :

1. $\dot{x}^i = f^i(x^1, \ldots, x^n, u^1, \ldots, u^n), \quad i = 1, \ldots n$ with $f(x,u) \in C^1$ in $R^n \times \omega$, where Ω will constitute a certain family of measurable m-vector functions.

2. Let X_0 and $X_1 \subset R^n$ be given as initial and target sets and let the nonempty control restraint $\Omega \subset R^M$.

3. The class $\triangle \subset \Omega$ is the class of all measurable controllers u with response $x(t, x_0)$ which steer x_0 to $x(t_1, x_0) = x_1 \in X_1$.

4. The cost functional is $C(u) = \int_0^{t_1} f^0(x(t), u(t))dt$ where $f^0(x,u) \in C^1$ in $R^n \times \Omega$.

An admissible controller $\bar{u}(t)$ is **minimal (optimal),** if $C(\bar{u}) \leq C(u)$ for all $u \in \Omega$.
Let us now define:

- The augmented state $\bar{x}^*(t) = \begin{pmatrix} x^{0*}(t) \\ x^*(t) \end{pmatrix}$ as the response to the augmented system:

$$\begin{array}{l} \dot{x}^0 = f^0(x,u) \\ \dot{x}^i = f^i(x,u), \quad i = 1, ..., n. \end{array} \qquad (\bar{\mathcal{S}})$$

- Let $\bar{\eta}^*(t)$ be a nontrivial solution of the augmented adjoint system [6]:

$$\begin{array}{l} \dot{\eta}^0 = 0 \\ \dot{\eta}^i = -\sum_{j=0}^{n} \eta_j \frac{\partial f^j}{\partial x^i}(x^*(t), u^*(t)) \quad i = 1, ..., n \end{array} \qquad (\bar{\mathcal{A}})$$

  Where the last n equations form the adjoint system $(\mathcal{A})$.

- And define the augmented Hamiltonian:

$$\bar{H}(\bar{\eta}, \bar{x}, \bar{u}) = \eta_0 f^0(x,u) + H(\eta, x, u)$$

  and

$$\bar{M}(\bar{\eta}, \bar{x}) = \max_{u \in \Omega} \bar{H}(\bar{\eta}, \bar{x}, u)$$

  Here the Hamiltonian function is:

$$H(\eta, x, u) = \eta f(x,u) = \eta_1 f^1(x,u) + ... + \eta_n f^n(x,u)$$

[6] Observe that this condition is equivalent to to the next one:$(\eta_0, \eta(t)) \neq 0 \ \ \forall t \in [0, t-1]$.

Theorem 54 *Consider the autonomous control process in* R^n

$$\dot{x} = f(x,u) \qquad (\mathcal{S})$$

- *Let* X_0 *and* $X_1 \subset R^n$ *be given the initial and target sets and let* $\bigtriangledown$ *be the set of all admissible controllers that steer some initial point of* X_0 *to a final point in the target set* X_1.
- *The terminal time* t_1, *the initial point* $x_0 \in X_0$ *and the terminal point* $x_1 \in X_1$ *vary with the control.*
- *For each* $u(t)$ *with response* $x(t)$ *let us assign a cost*

$$C(u) = \int_0^{t_1} f^0(x(t), u(t))dt$$

- *with* $f(x,u), f^0(x,u)$, $\partial f/\partial x(x,u)$ *and* $\partial f^0/\partial x(x,u)$ *continuous in* R^{n+m}.

If $u^*(t)$ *on* $0 \le t \le t^*$ *is a minimal optimal in* $\bigtriangledown$, *with augmented response* $\bar{x}^*(t) = x^{0*}(t), x^*(t))$ *then there exists a nontrivial augmented adjoint response* $\bar{\eta}^*(t) = (\eta_0^*, \eta^*(t))$ *such that*

$$\bar{H}(\bar{\eta}^*(t), \bar{x}^*(t), \bar{u}^*(t)) = \bar{M}(\bar{\eta}^*(t), \bar{x}^*(t))$$

and

$$M(\bar{\eta}^*(t), \bar{x}^*(t)) \equiv 0 \;\; \textit{and} \;\; \eta_0 < 0 \;\; \textit{everywhere.}$$

That is there exists a nontrivial adjoint response $\bar{\eta}(t)$ *of*

$$\dot{\eta} = -\eta \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t)) \qquad (\mathcal{A})$$

such that the maximal principle obtains, that is,

$$\bar{M}(\bar{\eta}, \bar{x}) = \max_{u \in \Omega} \bar{H}(\bar{\eta}, \bar{x}, \bar{u}) \;\; \textit{almost everywhere.}$$

And the following transversal conditions:

9.3 Transversality conditions

Also if the if X_0 and X_1 (or just one of them) are manifolds with tangent spaces, T_0 and T_1 at $x^*(0)$ and $x^*(t^*)$ then $\bar{\eta}^*(t) = (\eta_0^*(t), \eta^*(t))$ can be selected to satisfy the *transversality conditions* at both ends, or just one end:

$$\bar{\eta}^*(0) \;\; \text{orthogonal to} \;\; T_0$$

$$\bar{\eta}^*(t^*) \;\; \text{orthogonal to} \;\; T_1$$

Remark 55 *If the target set* X_1 *is the whole* R^n*, then the control problem is known as the* **free-endpoint problem.** *If the optimal control and its associate response satisfy the maximal principle, then the final transversality condition requires* $\eta^*(t_1) = 0$*.*

Remark 56 *If in the control problem the time interval is fixed, this problem is known as the* **fixed-time problem,** *then there exists a nontrivial adjoint response* $\bar{\eta} = (\eta_0, \eta)$ *on* $0 \leq t \leq T$ *such that*

$$\bar{H}(\bar{\eta}^*(t), \bar{x}^*(t), u^*(t)) = \bar{M}(\bar{\eta}^*(t), \bar{x}^*(t))$$

almost everywhere, and $\bar{M}(\bar{\eta}^*(t), \bar{x}^*(t))$ *is constant and* $\eta_0 \leq 0$*. The transversality conditions are verified, but as no time perturbations are allowed that is* $\bar{K}^{\mp}$ *is replaced by* $\bar{K}_t$ *we are unable to maintain the vanishing of* $\bar{M}(\bar{\eta}^*(t), \bar{x}^*(t))$

Proof: Let $u^*(t)$ on $0 \leq t \leq t^*$, with response $x^*(t)$ steering from $x^*(0) = x_0^* \in X_0$ to $x^*(t^*) = x_1^* \in X_1$, be optimal in $\triangle$. Consider the augmented system in R^{n+1}

$$\dot{x}_0 = f^0(x_1, ..., x_n, u)$$

$$\dot{x}_i = f_i(x_1, ..., x_n, u) \quad i = 1, 2, ..., n.$$

or

$$\dot{\bar{x}} = \bar{f}(x, u) \qquad (\mathcal{S}),$$

with corresponding response $\dot{x} = (x_0^*(t), x^*(t))$ where

$$x_0^*(t) = \int_0^t f^0(x^*(s), u^*(s))ds.$$

Each control $u(t) \in \triangle$ determines some augmented response $\bar{x}(t)$, leading from X_0 to X_1. The optimal controller steers the x_0 to the lowest possible point en $R \times x_1^*$. Then using the PMP, there exists a nontrivial adjoint response $\bar{\eta}^*(t) = (\eta_0^*, \eta^*(t))$ so,

$$\bar{H}(\bar{\eta}^*, \bar{x}^*, \bar{u}^*) = M(\bar{\eta}^*, \bar{x}^*)$$

almost everywhere, and

$$M(\bar{\eta}^*, \bar{x}^*) = \bar{M}$$

is constant everywhere on $0 \leq t \leq t^*$.

We now prove that $\bar{M} = 0$. (Plan of the proof, the details in [Lee, E.; Markus, L.] 313. This result follows as a consequence of the minimizing property of $u^*(t)$. Define the perturbation cone $\bar{K}_\tau^{\pm}$, as the smallest closed cone at τ, $0 \leq \tau \leq t^*$ in the tangent space at $\bar{x}^*(\tau)$ containing the perturbation tangent cone $\bar{K}\tau$ and the two vector $v_+(\tau) = \bar{f}(x^*(\tau), u^*(\tau))$ and $v_-(\tau) = -\bar{f}(x^*(\tau), u^*(\tau))$. Any vector w in the interior of this cone, must define a line segment from $\bar{x}^*(\tau)$, which lies interior to $\cap_{0\le t\le t^*}\bar{K}(t)$. In particular $w_\tau = (-1, 0)$ does not lie interior to $\bar{K}_\tau^\pm$, other wise there exists an admissible controller of total cost less than $C(u^*) = x^{0*}(t)$. So, $\bar{K}_\tau^\pm$ is separated from w_τ from an hyperplane with a normal vector $\bar{\eta}_\tau(\tau)$. Let $\bar{\eta}^*(t)$ the solution of the adjoint augmented equation, Then from the maximal principle

$$\bar{H}(\bar{\eta}^*(t), \bar{x}^*(t), u^*(t)) = \bar{M}(\bar{\eta}^*(t)\bar{x}^*(t)) = \bar{M} \ \ (constant)$$

almost everywhere. Since $\bar{\eta}_\tau^* v\pm(\tau) \le 0$, and $v_(\tau) = v_-(\tau)$, we conclude $\bar{\eta}_\tau^* v\pm(\tau) \le 0$ and thus, $\bar{M} = 0$. This construction is valid at each Lebesgue time, the almost everywhere. In order to see that $\bar{M} \equiv 0$ everywhere we need to construe the limit cone $\bar{K}_t$. The details of this construction are in the above cited reference.

Finally we must select $\bar{\eta}^*$ such that $\bar{\eta(t)}$ satisfies the transversality conditions. Let T_0 be the tangent space of X_0 at x_0 and let $\bar{T}_0$ be the linear space at $(0, x_0^*)$ spanned by the vector of the form $(0, T_0)$. Similarly let $\bar{T}_1$ be all vectors $(0, T_1)$ at $(x^{0*}(t^*), x_1^*.)$ Let $\mathcal{K}_t$ be the smallest closed cone in the tangent space at $\bar{x}^*(t)$ generated by the displacement of $\bar{T}_0$ and $\bar{K}_t^\pm$. Let $\mathcal{T}_1$ be the cone in the tangent space generated by $w_{t^*} = (-1, 0)$ and $\bar{T}_1$. Suppose that the cones $\mathcal{K}_{t^*}$ and $\mathcal{T}_1$ are separated by an hyperplane π. In this case take a normal vector $\bar{\eta}^* = (\eta^(0*), \eta^*)$ at $\bar{x}^*(t^*)$ with $\eta_0 < 0$ and

$$\bar{\eta}^*\mathcal{K}_{t^*} \le = \quad \bar{\eta}^*\mathcal{T}_1 \ge 0.$$

The linear space $\bar{T}_1$ which lies in $\mathcal{T}_1$ must lies in the hyperplane π. Thus the vector η^* satisfies the transversality condition. From parallel displaced $A_{t^*0}\bar{T}_1$ it follows that $\eta^*(0)\bar{T}_0 = 0$.

The entire proof will be complete justifying the separation of $\mathcal{T}_1$ and $\mathcal{K}_t$. See above cited reference.

9.4 A resource allocation problem

In this section we shall illustrate how the maximal principle and the existence theorem are used to find the optimal control. Consider the following resource allocation problem

Example 57 *Let Y denote the rate of production at time t of a certain commodity. Let $I(t)$ be the rate of investment of this commodity, and let C denote the rate of consumption. The equilibrium equation is given by: $Y = I + C$, $I \ge 0$, $C \ge 0$. Assume that $Y(t) = Y_0 + \int_0^t I(s)ds$. The first equality say that all the commodity produced in a given period is allocated to investment or consumption. The second equality, reflects the assumption that the commodity allocated to* *investment is used to increase the stock of the commodity. The constant Y_0 is the initial capacity of production. Let $T > 0$ be given. The production planner is to choose at each instant t the magnitudes $I(t)$ and $C(t)$ so that $U(C) = \int_0^T C(s)ds$ is maximized.*

Let us now formulate this problem as a control problem. Let $u(t)$ denote the fraction of the commodity produced in time t that is allocated to investment. Thus: $0 \leq u(t) \leq 1$, and $1 - u(t)$ is the fraction allocated to consumption. Hence

$$I(t) = u(t)Y(t)$$

$$C(t) = (1 - u(t))Y(t)$$

The preceding problem is equivalent to the following control problem: Minimize

$$C(u) = -\int_0^T (1 - u(s))Y(s)ds$$

subject to

$$\frac{dY}{dt} = u(t)x \quad Y(0) = Y_0$$

$$0 \leq u(t) \leq 1, \quad Y \geq 0.$$

where $Y_0 > 0, T$ is fixed, and the terminal state Y_1 is non negative, but otherwise arbitrary. In this case we have that:

$$\Omega = \{u : \text{ is measur. and } 0 \leq u(t) \leq 1\} \qquad f^0(Y, u, t) = -(1 - u)Y(t) \quad \text{and} \quad f(Y, u, t) = uY.$$

Exercise 8 1. *Show that the hypothesis of the problem satisfy the hypotheses of the existence theorem.*

2. *Show that $Y_0 \leq Y(t) \leq Y_0 e^t$, hence the condition $y(t) \geq 0$ is always omitted, and so it can be omitted from further considerations.*

3. *From the maximal principle determine the optimal pair (y^*, u^*). In order to do this observe that the maximal principle becomes:*

   - $[\eta_0 + \eta(t)Y^*(t)]\, u^*(t) \geq [\eta_0 + \eta(t)Y(t)]\, u$ *for all* $u \in [0, 1]$.
   - *Show that the transversality condition is $\eta(T) = 0$ (recall that the objective is $Y(T) \geq 0$.) This implies $\eta_0 < 0$.*
   - *Show that $u = sup\,\{0, sgn(-1 + \eta(t))\}$ Then $u(T) = 0$.*
   - *Consider $\delta > 0$ such that $[T - \delta, T]$ is the maximal interval such that $u(t) = 0$ and show that $\delta = 1$.*

- *Consider $T > 1$. Show that* $\eta(t) = \begin{cases} < 1 & t \geq T-1 \to u(t) = 0 \\ = 1 & t = T-1 \\ > 1 & t < T-1 \to u(t) = 1. \end{cases}$

4. *Finally:*

$$Y^*(t) = \begin{cases} Y_0 e^t & if \;\; 0 \leq t \leq T-1 \\ \\ Y_0 e^{T-1} & if \;\; T-1 \leq t \leq T. \end{cases} \quad and \quad u^*(t) = \begin{cases} 1 & if \;\; 0 \leq t \leq T-1 \\ \\ 0 & if \;\; T-1 \leq t \leq T. \end{cases}$$

The procedure used in the preceding example is one that can often be used. In large scale problems, obtaining an analytical solution is not easy, we use in these cases computational methods.

9.5 Infinite time duration

If the fixed time duration becomes infinite we obtain an interesting problem. Consider fixed initial-point x_0 and end-point x_1, measurable controllers $u(t)$ on $0 \leq t < \infty$, bounded on each compact time interval, satisfying the restraint Ω and each of which defines a response $x(t)$ on $0 \leq t < \infty$, with $\lim_{t\to\infty} = x_1$ in R^n. The set $\triangle_\infty$ of all admissible controllers consist of all such controllers for which the cost is convergent.

The condition $\bar{M}(\bar{\eta}^*(t), \bar{x}^*(t)) = 0$ is verified for all $0 \leq t < \infty$ as the following theorem assert.

Theorem 58 *Consider the control process in R^n*

$$(\mathcal{S}) \qquad \dot{x} = f(x, u).$$

The measurable controllers $u(t) \subset \Omega$ on $0 \leq t < \infty$, with responses $x(t)$ steering x_0 to x_1, having finite cost:

$$C(u) = \int_0^\infty f^0(x(t), u(t))dt,$$

belong to the admissible class $\triangle_\infty$. Let $u^*(t)$, with augmented response $\bar{x}(t)$, be optimal in $\triangle_\infty$. Then there exists a nontrivial augmented adjoint response $\bar{\eta}^* = (\eta_0^*, \eta^*(t))$ such that

$$\bar{H}(\bar{\eta}^*(t), \bar{x}^*(t), \bar{u}^*(t)) = \bar{M}(\bar{\eta}^*(t), \bar{x}^*(t)) \;\; everywhere \quad on \; 0 \leq t < \infty, .$$

and

$$M(\bar{\eta}^*(t), \bar{x}^*(t)) \equiv 0 \;\; everywhere \quad on \; 0 \leq t < \infty, .$$

and $\eta^{*0} \leq 0$.

Proof: For each time interval $0 \leq t \leq T$ consider the class Δ_T of bounded measurable controllers in Ω that steer x_0 to $x^*(T)$. Then $u^*(t)$ is optimal in Δ_T, for otherwise any smaller cost controller in Δ_T could be supplemented by $u^*(t)$ on $T \leq t \leq \infty$, contradicting the optimality of $u^*(t)$ in Δ_∞.

Let $\bar{\eta}^*(T) = (\bar{\eta}_T^{*0}, \bar{\eta}_T^*(t))$ be an adjoint response to

$$\dot{\bar{\eta}} = \frac{\partial \bar{f}}{\partial \bar{x}}(x^*(t), \eta^*(t))$$

such that:

$$\bar{H}(\bar{\eta}_T^*(t), \bar{x}^*(t), \bar{u}^*(t)) = \bar{M}(\bar{\eta}_T^*(t), \bar{x}^*(t)) \ \ everywhere \ \ \ on\ 0 \leq t \leq T$$

and

$$M(\bar{\eta}_T^*(t), \bar{x}^*(t)) \equiv 0 \ \ and \ \ \eta_0 < 0 \ \ everywhere \ \ \ on\ 0 \leq t < T.$$

Also $\eta^{*0} \leq 0$, and we can choose $\bar{\eta}_T(0)$ to be a unit vector. The convergence is uniform on compact intervals.

Now let $T = 1, 2, ...$ and select a convergent subsequence of a unit vector :

$$\lim_{T\to\infty} \bar{\eta}_T^*(t) = \bar{\eta}^*(t) \ on\ 0 \leq t \leq \infty.$$

Suppose

$$\bar{H}(\bar{\eta}^*(t), \bar{x}^*(t), \bar{u}^*(t)) < \bar{M}(\bar{\eta}^*(t), x^*(t))$$

on some set of positive duration, then for sufficiently large T, we obtain a contradiction with the last theorem. The proof of theorem (54), shows that

$$M(\bar{\eta}_T^*(t), \bar{x}^*(t)) \equiv 0 \ \ on\ 0 \leq t < \infty.$$

9.6 Example: A capital accumulation model

Consider a firm that needs capital goods to produce commodities, which are sold on the output market. The more capital goods the firm owns, the more commodities it can sell and thus more revenue R is obtained. The firm can increase its capital stock K by investment, the investment rate is denoted by I. We assume that there exist investment cost $c(I)$ which are assumed to be represented by a convex function. Changes in the investment imply changes in the organization of the firm and these changes have costs.. Representing the changes in investment by v these costs are equal to $g(v)$; we assume that g is a convex function.

As usual the following equation for capital stock arises:

$$\dot{K} = I - \delta K$$

assuming constant rate of depreciation δ. We assume that $R(K)$ is positive.

Example 59 *The firm's objective is to maximize the discounted cash flow over an infinite planning horizon. Summarizing we obtain the following model:*

$$\min_v -\int_0^\infty [R(K) - C(I) - g(v)]e^{-rt}dt.$$

$$\begin{aligned} s.t.\quad & \dot{K} = I - \delta K \\ & \dot{I} = v. \\ & R(K) > 0, \quad I(0) = I_0, \quad K(0) = K_0. \end{aligned}$$

In this case K and I are the state variables, and v is the control variable. The control process can be written like:

$$\begin{bmatrix} \dot{K} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -\delta & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} K \\ I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix}$$

That is $\dot{x} = Ax + Bu$, where $x = (K, I)$ and

$$A = \begin{bmatrix} -\delta & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Here $f^0(K, I, t) = [R(K) - C(I)]e^{-rt}$ and $h^0(u, t) = (v)e^{-rt}$ then the adjoint system associate with our problem is:

$$\dot{\eta} = \frac{\partial f^0}{\partial x}(t, x) - \eta A$$

So, we need to solve

$$\max_v \left[g(v)e^{-rt} + \eta B u \right],$$

where $\eta = (\eta_1, \eta_2)$ and $u = (v, 0)$.

This problem is equivalent to the following one

$$\max_v \left[g(v)e^{-rt} + \eta_2 v \right].$$

So, $\eta_2 = -g'(v)e^{-rt}$.

To analyze examples involving a discount factor, it is possible to use the current-value hamiltonian, $\mathcal{H}$ in lieu of H. By introducing the new variables $m_i = \eta_i e^{rt}$, where r is the discount rate we can introduce the current-value version of H as follows:

$$\mathcal{H} = He^{rt} = \eta_0 f^0(x, t, u) + mf(x, t, u).$$

In this case we obtains, $\eta_0 < 0$ and

$$\mathcal{H} = [R(K) - C(I) - g(v)] + m_1(I - \delta K) + m_2 v.$$

Maximization of the hamiltonian with respect to the control variable v is equivalent to the following maximization problem:

$$max_v \, [g(v) + m_2 v]$$

Assuming a quadratic cost function is quite common in the literature, so consider $g(v) = \frac{\alpha}{2}v^2$.

Further application of Pontriaguin Maximun Principle leads to the following differential dynamic system:

$$\begin{array}{l} \dot{K} = I - \delta K \\ \dot{I} = v \end{array} \tag{22}$$

$$\begin{array}{l} \dot{m}_1 = -R'_K(K) + (\delta + r)m_1 \\ \\ \dot{m}_2 = C'_I(I) - m_1 + rm_2 \end{array} \tag{23}$$

The steady state. From $\dot{K} = 0, \;\; \dot{I} = 0$, it is straightforward to see why $I = \delta K$ and $v = 0$ is required for the steady state. Additional to these equations $\dot{m}_1 = 0, \;\; \dot{m}_2 = 0$, imply that

$$m_1 = C'_I(\delta K) = \frac{R'_K(K)}{\delta}.$$

Exercise 9 *This exercise is inspired by [Haunschmied, J.; Kort, P.; Hartl, R.; Feichtinger, G.].*

1. *Is it a process controllable?*

2. *Show that in case of a convex cost function $C(I)$ and a the revenue function $R(K)$ that is concave, there exists at most one steady state.*

3. *Analyze the possibility of the existence of more than one steady state depending on the properties of $R(K)$ and $C(I)$*

4. *Suppose that $C(I) = aI + bI^2$, and analyze the possibility that there exist more than one steady state. Analyze the stability of the stationary states according to the properties of the function R. In order to do this, it can be useful to analyze the behavior of the jacobian of the differential system given by:*

$$\begin{array}{l} \dot{K} = I - \delta K \\ \dot{I} = \frac{m_2}{\alpha} \\ \dot{m}_1 = -R'_K(K) + (\delta + r)m_1 \\ \dot{m}_2 = C'_I(I) - m_1 + rm_2 \end{array}.$$

5. *Analyze the situation when* $C(I) = aI + bI^2$, $R(K) = -K^2 + K + c$ *and* $g(v) = \frac{\alpha}{2}v^2$, *here* a, b, c *and* α *are real numbers. Describe the optimal trajectories depending of the initial conditions.*

10 The maximal principle for nonlinear non-autonomous control process

We now turn to the most general nonlinear non-autonomous control process. We shall show here that the maximal principle is also in this case a necessary condition for optimality.

The maximal principle for such processes will be obtained as an immediate consequence of the theorems for the existence in non-autonomous processes, by introducing the time a s a new coordinate $x^{n+1} = t$.

In the following analysis we assume:

1. $(\mathcal{S})$ $\dot{x} = f(x, t, u)$ is a control processes in R^n with $f \in C^1(R^{n+1+m})$.

2. The initial and target sets X_0 and X_1 are nonempty in R^n.

3. The admissible controllers Δ are bounded measurable functions $u(t)$ on various finite time intervals $t_0 \leq t \leq t_1$ satisfying some restraint, $u(t) \in \Omega \subset R^n$, and each steering some point $x_0 \in X_0$ to some point $x_1 \in X_1$.

4. The cost of controller $u(t)$ on $t_0 \leq t \leq t_,$ in Δ with response $x(t)$ is:

$$C(u) = \int_{t_0}^{t_1} f^0(x(t), t, u(t))dt$$

   where $f_0 \in C^1(R^{n+1+m})$.

The time augmented response to $u(t)$ is:

$$\tilde{x}(t) = (x_0(t), x(t), x^{n+1}(t)),$$

which is the solution of:

$$(\tilde{\mathcal{S}}) \qquad \dot{\tilde{x}} = \tilde{f}(\tilde{x}, u)$$

or

$$\dot{x}_0 = f^0(x, x^{n+1}, u)$$

$$\dot{x} = f(x, x^{n+1}, u)$$

$$\dot{x}^{n+1} = 1$$

with $\tilde{x}(t_0) = (0, x_0, t_0)$.

The time-augmented adjoint system, based on $u(t)$ and $\tilde{x}(t)$, is

$$(\tilde{\mathcal{A}}) \qquad \dot{\tilde{\eta}} = -\tilde{\eta}\frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x}(t), u(t))$$

or

$$\dot{\eta}_0 = 0$$

$$\dot{\eta}_j = -\sum_{i=0}^{n} \eta_i \frac{\partial f^i}{\partial x^j}(x(t), t, u(t)) \quad j = 1, ..., n$$

$$\dot{\eta}_{n+1} = -\sum_{i=0}^{n} \eta_i \frac{\partial f^i}{\partial t}(x(t), t, u(t))$$

The time augmented Hamiltonian function is:

$$\tilde{H}(\tilde{\eta}, \tilde{x}, u) = \eta_0 f^0(x, x^{n+1}, u) + ... + \eta_n f^n(x, x^{n+1}, u) + \eta_{n+1} \ \ \textit{and}$$

$$\tilde{M}(\tilde{\eta}, \tilde{x}) = \max_{u\in\Omega} \tilde{H}(\tilde{\eta}, \tilde{x}, u).$$

We also write

$$\tilde{x} = (\bar{x}, x^{n+1}), \qquad \tilde{\eta} = (\bar{\eta}, \eta_{n+1})$$

$$\tilde{H}(\tilde{\eta}, \tilde{x}, u) = \bar{H}(\bar{\eta}, \bar{x}, t, u) + \eta_{n+1}$$

$$\tilde{M}(\tilde{\eta}, \tilde{x}) = \bar{M}(\bar{\eta}, \bar{x}, t) + \eta_{n+1}.$$

Now we show the existence of optimal controllers for non linear non autonomous processes when the restraint set Ω is compact. We consider the case where the initial and target sets are nonempty compact sets at each time varying continuously, and we consider state constraints.

Theorem 60 *Consider the nonlinear process in* R^n

$$(\mathcal{S}) \qquad \dot{x} = f(x, t, u) \ \ in \ \ C^1(R^{n+m+1}),$$

the data are as follows:

The initial and target sets $X_0(t), X_1(t)$, *are nonempty compact set at each time varying continuously, for all* t *in a prescribed compact interval* $[t_0, t_1]$.

The control restraint set $\Omega(x, t) \subset R^m$ *is a nonempty compact set at each time varying continuously,* $(x, t) \in R^n \times [t_0, t_1]$.

The state constraints $h^1(x) \geq 0, ...h^r(x) \geq 0$, *are real continuous functions on* R^n.

The family Δ of admissible controllers consists of all measurable functions on various time intervals $[t_0, t_1]$ which associate response steering $x(t_0) \in X(t_0)$ to $x(t_1) \in X(t_1)$ and $u(t) \in \Omega(x,t)$.

The cost functional for each $u \in \Delta$ is

$$C(u) = g(x(t_1)) + \int_{t_0}^{t_1} f^0(x(t), t, u(t))dt + \max_{t_0 \le t \le t_1} \gamma(x(t))$$

where $f^0 \in C^1(R^{n+m+1})$, and $g(x)$ and $\gamma(x)$ are continuous in R^n.

Assume

- *(a) There exists a uniform bond $|x(t)| \le b$ on $[t_0, t_1]$.*
- *(b) The extended velocity set $\tilde{V}(x,t) = \{f^0(x(t), t, u(t)), f(x(t), t, u(t)) : u \in \Omega(x,t)\}$ is convex on R^{n+1} for each fixed (x,t).*

Then there exists an optimal controller $u^*(t)$ on $t_0 \le t \le t_1$ in Δ minimizing $C(u)$.

Proof: The proof of this theorem is given in [Lee, E.; Markus, L.] 260. (Idea of the demonstration) Since $\Omega(x,t)$ lies within a bounded set in R^m and $|x| \le b$ all $u(t) \in \Delta$ and $x(t)$ are uniformly bounded. Thus there is a finite lower bound for the costs of admissible controllers. Choose now a sequence of controllers $u_k(t) \in \Delta$ with $C(u_k)$ decreasing monotonically to $inf C(u) \;\; \forall u \in \Delta$. Now we select a subsequence $u_{k'}$. We must show that this subsequence leads to an admissible controller $u^* \in \Delta$ realizing the minimun cost. To prove this we need the Ascoli's theorem and the weak convergence of $u_k(t) \in \Omega(x,t)$.

Theorem 61 *Consider the process in R^n $(\mathcal{S})$ $\;\; \dot{x} = f(x,t,u)$.*

Let Δ be all bounded measurable controllers $u(t) \in \Omega \subseteq R^n$, on various finite time intervals $t_0 \le t \le t_1$, steering some point $x_0 \in X_0$ to some point $x_1 \in X_1$ as above with cost

$$c(u) = \int_{t_0}^{t_1} f_0(x(t), t, u(t))dt.$$

If $u^*(t)$ on $t_0^* \le t \le t_1^*$ with time-augmented response $\tilde{x}^*(t)$ is optimal in Δ, then there exists a non trivial time-augmented adjoint response $\tilde{\eta}^*(t)$ of $\tilde{A}$ such that:

$$\tilde{H}(\tilde{\eta}^*(t), \tilde{x}^*(t), u^*(t)) = \tilde{M}(\tilde{\eta}^*, \tilde{x}^*(t)) \quad a.e.$$

and

$$\tilde{M}(\tilde{\eta}^*, \tilde{x}^*(t)) \equiv 0, \;\; \eta_0 \le 0, \;\; \textit{everywhere on } t_0^* \le t \le t_1^*.$$

These conclusions can also be written

$$\bar{H}(\bar{\eta}^*(t), \bar{x}^*(t), t, u^*(t)) = \bar{M}(\bar{\eta}^*(t), \bar{x}^*(t), t); \quad a.e.$$

and

$$\bar{M}(\bar{\eta}^*(t), \bar{x}^*(t), t) \equiv \int_{t_0}^{t_1} \sum_{i=0}^{n} \eta_i \frac{\partial f_i}{\partial t}(x^*(s), s, u^*(s))ds.$$

The transversality conditions yield

$$\eta^*_{n+1}(t^*_0) = \eta^*_{n+1}(t^*_1) = 0,$$

so,

$$\bar{M}\bar{\eta}^*(t^*_1), \bar{x}^*(t^*_1), t^*_1) = 0.$$

If X_0 and X_1 (or just one of them) are manifolds in R^n with tangent spaces T_0, and T_1 at x^*_0 and x^*_1, respectively, then $\bar{\eta}^*(t)$ can be selected to satisfy the further conditions (or at just one them)

$$\eta^*(t^*_0) \;\; transv \; T_0, \;\; \eta^*(t^*_1) \;\; transv \; T_1.$$

Proof: In the space R^{n+1} of (x, x^{n+1}) the control problem:

$$\dot{x} = f(x, x^{n+1}, u)$$

$$\dot{x}^{n+1} = 1$$

with cost:

$$C(u) = \int_{t_0}^{t_1} f^0(x(t), x^{n+1}, u(t))dt$$

is an autonomous process as considered before. The initial and target sets are cylinders $X_0 \times R^1$ and $X_1 \times R^1$. Since $\dot{x}^{n+1} = 1$, each controller $u(t)$ of this autonomous problem steers (x_0, t_0) to (x_1, t_1).

From the theorem for the autonomous process we obtain the necessary conditions

$$\tilde{H}(\tilde{\eta}^*(t), \tilde{x}^*(t), u^*(t)) = \tilde{M}(\tilde{\eta}^*, \tilde{x}^*(t)) \quad a.e.$$

and

$$\tilde{M}(\tilde{\eta}^*, \tilde{x}^*(t)) \equiv 0, \;\; \eta_0 \leq 0, \;\; everywhere \; on \; t^*_0 \leq t \leq t^*_1.$$

The assertions $\bar{H} = \bar{M}$ and

$$\bar{M} = \int_{t_0}^{t_1} \sum_{i=0}^{n} \eta_i \frac{\partial f_i}{\partial t}(x^*(s), s, u^*(s))ds$$

follow directly from the definitions preceding this theorem and the calculation

$$\eta^*(t_1) = -\int_{t_0}^{t_1}\sum_{i=0}^{n}\eta_i\frac{\partial f_i}{\partial t}(x^*(s), s, u^*(s))ds + \eta^*_{n+1}(t^*_0).$$

Transverslaty conditions follows directly. In fact $\tilde{\eta}^*(t)$ can be chosen in such a way that $(\eta^*(t_i), \eta_{n+1}(t_i))$ is ortogonal to the line $x^*_i \times R^1 \;\; i = 1, 2$. This means that $\eta_{n+1}(t_i) = 0,\ i = 1, 2$.

Remark 62 *1. For the nonoautonomous problem but fixed initial t_0 allowing various final times $t > t_1$, the necessary conditions and transversality conditions are the same, with the exception that we can not assert that $\eta^*_{t_0}$ vanishes.*

2. We consider now the case of varying initial and target sets.

The maximal principle holds as before. $\tilde{H} = \tilde{M}$ and $\tilde{M} = 0$ with $\eta_0 \leq 0$. Again implies $\bar{H} = \bar{M}$ and $\bar{M} = \int_{t_0}^{t_1}\sum_{i=0}^{n}\eta_i\frac{\partial f_i}{\partial t}(x^*(s), s, u^*(s))ds$

The transversality conditions assert that:

$$(\eta^*(t_i), \eta_{n+1}(t_i)) \quad \textit{is transversal to } X_i(t^*_i) \textit{ at } (x^*_i, t_i) \;\; i = 1, 2, \textit{ in } R^{n+1}.$$

If t_i is fixed only the transversal condition at t_j is fulfilled, $i \neq j, i, j \in \{1, 2\}$.

The transversality conditions can be written as:

$$\eta^*(t^*_i)q_i + \eta^*_{n+1}(t^*_i) = 0 \;\; i = 1, 2.$$

where $q_i = \dot{x}_i(t^*_i)$ is the velocity of the target (initial) point. In this case

$$\bar{M}(\bar{\eta}^*(t_i(, \bar{x}^*(t_i), t^*_i) = \eta^*(t^*_i)q_i.$$

11 Sufficient condition for an autonomous process in R^n.

We now turn to sufficiency conditions for an optimal control

Consider a control precess in R^n, $\quad \dot{x} = f(x, t, u)$. The cost functional is

$$C(u) = g(x(T)) + \int_{t_0}^{T} f^0(x(t), t, u(t))dt$$

where g, f, f^0 are in class C^1. The admissible controllers $u(t)$ are all bounded measurable $m-vector$ functions on the fixed interval $[t_0, T]$ steering x_0 to some point in $\mathcal{T}$ and lying in some nonempty restraint set $\Omega \subset R^n$.

In the next two subsections we use a dynamic argument to derive the maximal principle. Although the arguments are mathematically correct, the assumptions are such that most interesting problems are ruled out. The purpose of these subsections is to give some insight to the concept of the feedback control or optimal synthesis.

In the third subsection we shall give a sufficient criterium for optimality.

11.1 The method of dynamic programming

We shall seek a method for constructing a successful control for each initial state which can be steered to the target, by a well determined pair $(x(s), u(s))$, $t_0 \leq s \leq T$ in such way that for some function $u^0(\eta, s, x) = u(s)$ maximize $H(\eta, x, t, u)$ for each fixed (η, x, t).

It is often said that in this case a synthesis has been effected in $\mathcal{C}$, and that $u^0(\eta, s, x)$ is a feedback control function in $\mathcal{C}$.

Definition 63 *The control process in R^n, $\dot{x} = f(x, t, u)$ with restraint set $\Omega \in R^m$ and the hamiltonian function:*

$$H(\eta, x, t, u) = -f^0(x, t, u) + \eta f(x, t, u).$$

has a **feedback control** $u^0(\eta, x, t)$ *in case:*

$$H^0(\eta, x, t) = \max_{u \in \Omega} H(\eta, x, t, u) = H(\eta, x, t, u^0(\eta, x, t)).$$

We shall seek a feedback control $u^0(\eta, x, t)$ that maximizes

$$H(\eta, x, t, u) = -f^0(x, t, u) + \eta f(x, t, u),$$

for each $(\eta, x, t) \in R^{n+n+1}$

For each controller $u(t)$ on $[t_0, T]$ with corresponding response $x(t)$ and and steering x_0 to $x(T) \in G$. The cost is

$$C(u) = \int_{t_0}^{t_0+\delta} f^0(x, t, u)dt + \int_{t_0+\delta}^{T} f^0(x, t, u)dt$$

where $\delta > 0$ is an arbitrarily small number. Suppose that the minimal cost is; $V(x_0, t_0)$ and $V(x, t)$ is C^2 for $x \in R^n$ and $t_0 \leq t \leq T$ then the method of dynamic programming asserts that:

$$V(x_0, t_0) = \min_{u \in \Omega} \left\{ \int_{t_0}^{t_0+\delta} f^0(x, t, u)dt + V(x(t_0 + \delta), t_0 + \delta) \right\}$$

Expanding the above functions in terms of the small δ

$$V(x_0,t_0) = \min_{u\in\Omega}\left\{f^0(x_0,t_0,u)\delta + V(x_0.t_0) + \delta[V_x(x_0,t_0)f + V_t]\right\}.$$

This yields the functional equation for $V(x,t)$

$$-V_t(x,t) = \min_{u\in\Omega}\left\{f^0(x,t,u) + V_x(x,t)f(x,t)\right\}.$$

If we write $S(x,t) = -V(x,t)$ then

$$S_t = -\max_{u\in\Omega}[-f^0(x,t,u) + S_x f(x,t,u)]$$

or $S_t = -H^0(S_x,x,t)$. This equation is sometimes called the Bellman equation.

Thus the negative cost $S(x,t)$ satisfies the *Hamilton-Jacobi* partial differential equation:

$$\frac{\partial S}{\partial t} + H^0\left(\frac{\partial S}{\partial x}, x, t\right) = 0$$

with boundary data: $S(x,t) = -g(x)$ for $x \in G$.

11.2 The Hamilton-Jacobi equation

Theorem 64 *Assume that for the optimal control problem with the conditions given above, there exists a feedback control* $u^0(\eta,x,t)$ *in* C^1 *in* R^{n+n+1} $u^0(\eta,x,t)$ *such that:* $H^0(\eta,x,t) = H(\eta,x,t,u^0(\eta,x,t))$

(a) Let $S(x,t)$ *in* C^2 *for* $x \in R^n$ *and* $t \le T$ *be the solution of the Hamilton-Jacobi equation:*

$S_t + H^0(S_x,x,t) = 0$, *with* $S(x,T) = g(x)$ *for* $x \in G$.

Assume that the control law: $\bar{u}(x.t) = u^0(S_x(x,t),x,t)$ *determines a response* $\bar{x}(t)$ *steering* (x_0,t_0) *to* (G,T). *Then* $\bar{u}(t) = \bar{u}(\bar{x}(t),t)$ *is an optimal controller, with cost:*

$$C(\bar{u}(t)) = -S(x_0,t_0)$$

(b) On the other hand, assume that there exists an optimal controller for each initial state $x_0 \in R^n$ and arbitrary $t_0 \le T$ (T fixed), leading to the target set $\mathcal{T}$ with minimal cost $V(x_0,t_0) \in C^2$. Then $S(x,t) = -V(x,t)$ satisfies

$S_t + H^0(S_x,x,t) = 0$, with $S(x,T) = g(x)$ for $x \in G$.

Remark 65 *The existence of an appropriate solution* $S(x,t)$ *of the Hamilton-Jacobi equation, in a region* W *of the* (x,t) *space, is sufficient for the construction of a controller that is optimal among all those with responses in* W.

Proof: 350 [Lee, E.; Markus, L.].

Corollary 66 *Consider the autonomous process in R^n, $\dot{x} = f(x,u)$ with initial state x_0 and target $G = R^n$. The cost functional is*

$$C(u) = g(x(T)) + \int_{t_0}^{T} f^0(x(t), t, u(t))dt$$

where g, f, f^0 are in class C^1. The admissible controllers $u(t)$ are all bounded measurable $m-$ vectors functions on the fixed interval $[t_0, T]$ with values in the restraint set $\Omega \subset R^n$ Assume:

(a) There exists a feedback control $u^0(\eta, x)$ in $C^1 \in R^{n+n}$, which yields the unique point u^0 in Ω where:

$$H^0(\eta, x) = \max_{u \in \Omega}[-f^0 + \eta f] = H(\eta, x, u^0(\eta, x))$$

(b) Ω is either an open set or else the closure of an open set with C^1 smooth boundary in R^n.

Then an optimal controller $u^*(t)$ with response $x^*(t)$ is necessarily associate with and adjoint response $\eta^*(t)$ that satisfies the Hamiltonian system

$$\dot{x}_i = \frac{\partial H}{\partial \eta_i}, \quad \dot{\eta}_i = \frac{\partial H}{\partial x_i}, i = 1, ..., n$$

with boundary conditions $x^*(0) = x_0$, $\eta^*(T) = 0$ and that maximal principle

$$H^0(\eta^*(t), x^*(t)) = H(\eta^*(t), x^*(t), u^*(t))$$

holds almost everywhere in $0 \leq t \leq T$.

Proof: We know that the optimal controller $u^*(t)$ has response $x^*(t)$ and $\eta^*(t)$ satisfying:

$$\dot{x}_i = \frac{\partial H}{\partial \eta_i}(\eta, x, u^*(t)), \quad \dot{\eta}_i = -\frac{\partial H}{\partial x_i}(\eta, x, u^*(t)) \quad i = 1, ..., n.$$

Note that transversality conditions allow us to assume $\eta^* \equiv -1$ and $\eta(T) = 0$.

The maximal principle also holds:

$$H(u^*(t)) = \max_{u \in \Omega} H(\eta^*(t), x^*(t), u) = H^0(\eta^*(t), x^*(t)), \quad a.e.$$

so $u^*(t) = u^0(\eta^*(t), x^*(t))$.

We must show that $(x^*(t), \eta^*(t))$ is also a solution of the differential hamiltonian differential system specified by the function $H^0(\eta, x)$. For this compute the derivatives

$$\frac{\partial H}{\partial \eta}(\eta, x) = \frac{\partial H}{\partial \eta}(\eta, x, u^0) + \frac{\partial H}{\partial u}(\eta, x, u^0)\frac{\partial u^0}{\partial \eta}(\eta, x)$$

and

$$\frac{\partial H}{\partial x}(\eta, x) = \frac{\partial H}{\partial x}(\eta, x, u^0) + \frac{\partial H}{\partial u}(\eta, x, u^0)\frac{\partial u^0}{\partial x}(\eta, x)$$

If $u^*(t)$ lies in the interior to Ω then $(\partial H/\partial u) = 0$ at $(\eta^*(t), x^*(t), u) = (\eta^*, x^*, u^0((\eta^*, x^*))$, thus the corollary is proved.

Suppose now that a point $(\eta(t_1), x(t_1))$ is a limit of points (η, x) in R^{m+n} at which $u^0(\eta, x)$ lies in the interior of Ω then the corollary follows from the continuity considerations.

In other case, there exists a neighborhood N of $(\eta(t_1), x(t_1))$ such that for all point (η, x) in N $u^0(\eta, x)$ lies in the boundary of Ω : In this case observe that $\frac{\partial H}{\partial u}(\eta, x, u^0(\eta, x)$ is a normal vector to the boundary of Ω. And the vectors $\frac{\partial u^0}{\partial \eta}(\eta, x)$ and $\frac{\partial u^0}{\partial x}(\eta, x)$ are tangent ro the boundary of Ω. Then the corollary follows.

Remark 67 *Under the hypotheses of the corollary , the search for an optimal control is reduced to the solution of a nonlinear problem* $\max_{u\in\Omega} H(x, \eta, u)$ *for each fixed value of* x *and* η.

11.3 Sufficient condition for a locally optimal controller.

The sufficiency theorem for optimality will involve the second variation of the cost functional and will yield a local rather than global optimal controller.

The maximal principle for an autonomous control process in R^n

$$\dot{x} = f(x, u).$$

with initial state $x(0) = x_0$ and cost functional

$$C(u) = \int_0^T f^0(x, u)dt$$

with admissible controllers $u(t) \in \Omega$ are bounded and measurable functions on the fixed time finite interval $0 \le t \le T$,

$$H(\eta^*(t), x^*(t), u^*(t)) = \max_{u\in\Omega} H(\eta^*(t), x^*(t), u)$$

is a necessary condition for the optimality of $u^*(t)$, where the responses $x^*(t)$ and $\eta^*(t)$ satisfy:

$$\dot{x} = \tfrac{\partial H}{\partial \eta}(\eta, x, u^*(t))$$

$$\dot{\eta} = -\tfrac{\partial H}{\partial x}(\eta, x, u^*(t))$$

with $x(0) = x_0, \eta(T) = 0$. The hamiltonian function is here:

$$H(\eta^*(t), x^*(t), u^*(t)) = -f_0(x, u) + \eta f(x, u).$$

As we have seen the maximal principle, together with some convexity conditions on $f^0(x,u)$ and $f(x,u)$ yields a sufficient condition for an optimal controller $u^*(t)$. *We shall replace these global conditions by local convexity conditions* asserted in terms of second derivatives of f^0 and f and the we shall seek a sufficient condition for a locally optimal controller.

Definition 68 *A controller $u^*(t)$ is* locally optimal *in case there exists an $\epsilon > 0$ such that: for every admissible controller $u(t)$ with*

$$|u^*(t) - u(t)| \leq \epsilon \quad on \;\; 0 \leq t \leq T,$$

the cost is $C(u) \geq C(u^*)$.

Since we impose local convexity conditions, it is reasonable to assume that the candidate for optimality $u^*(t)$ lies everywhere in the interior of the restraint set Ω. Then the maximal principle asserts that:

$$\frac{\partial H}{\partial x}(\eta^*(t), x^*(t), u^*(t)) = 0.$$

Theorem 69 Sufficient conditions for optimal control. *As we have seen the maximal principle, together with some convexity hypotheses on f^0 and f yields a sufficient condition for an optimal controller. In the next theorem we replace these global convexity hypotheses by local convexity conditions, and we shall seek a sufficient condition for a locally optimal controller.*

Consider the autonomous process in R^n,

$$\dot{x} = f(x,u)$$

with initial state $x(0) = x_0$ and cost

$$C(u) = \int_0^T f^0(x,u)dt$$

where f and f^0 are in C^2 in R^{n+m}. The admissible controllers are each bounded and measurable functions $u(t)$ on the fixed interval $0 \leq t \leq T$ Let $u^*(t)$ be a controller interior to Ω and assume:

(1) $\frac{\partial H}{\partial u}(\eta^*(t), x^*(t), u^*(t)) = 0$ *almost always*
where: $H(\eta, x, u) = -f^0(x,u) + \eta f(x,u)$ *and*

(2) (η^*, x^*) *the* $(n+1)$ *vector* $\bar{\eta}(t) = (\eta_\alpha(t))$ *on* $0 \leq t \leq T$ *is the adjoint response, satisfy*

$$\dot{x}^\alpha = \frac{\partial H}{\partial \eta_\alpha} = f^\alpha(x(t), u(t)), \quad \alpha = 0, 1, ..., n.$$

$$\dot{\eta}_\alpha = -\frac{\partial \bar{H}}{\partial x^\alpha} = -\eta_0 \frac{\partial f^0}{\partial x^\alpha}(x, u(t)) - \ldots - \eta_n \frac{\partial f^n}{\partial x^\alpha}(x, u(t))$$

$$x(0) = x_0, \eta(T) = 0.$$

We define

(3)

$$f^0_{xx}p^2 + 2f^0_{xu}pq + f^0_{uu}q^2 \geq c(p^2 + q^2)$$

for arbitrary real constant n and m-vectors p and q and for a fixed constant $c > 0$, where these second partial derivatives are evaluated at almost every point $(x^*(t), u^*(t))$.

This implies that the symmetric matrix is positive definite

$$\begin{pmatrix} f^0_{xx} & f^0_{xu} \\ f^0_{xu} & f^0_{uu} \end{pmatrix} > 0$$

(4) Either of the following two conditions hold along $(x^*(t), u^*(t))$

(α)) $f^0_x = 0$.

(α) $f_{xx} = f_{xu} = f_{uu} = 0$

Then $u^*(t)$ is a locally optimal controller.

Note that $u(t)$ is called a maximal controller even though it yields a minimun cost.

Proof: Vary the controller $u^*(t)$ to $u(t) = u^*(t) + \epsilon\theta(t)$, The response $x(t)$ is then defined and it is easy to compute $|x^*(t) - x(t)| \equiv |\Delta x(t)| \leq k_1\epsilon$ where k_1 is a constant depending only on the given data. Also:
$\Delta x(t) = \int_0^t \left[\frac{\partial f}{\partial x}(x^*, u^*)(x^*(s) - x(s)) + \frac{\partial f}{\partial u}(x^*, u^*)\epsilon\theta(s)\right] ds$

$$+ \int_0^t \left[\frac{\partial^{\bar{2}} f}{\partial x^2)}\Delta x^2 + 2\frac{\partial^{\bar{2}} f}{\partial x \partial u}(\Delta x)(\epsilon\theta) + \frac{\partial^{\bar{2}} f}{\partial u^2}(\epsilon\theta)^2\right] ds.$$

Here the bar notation indicates that the second derivatives are evaluated at some point near $(x^*(s), u^*(s))$.

If we define $\psi(t)$ by:

$$\dot{\psi} = \frac{\partial f}{\partial x}(x^*, u^*)\psi + \frac{\partial f}{\partial u}(x^*, u^*)\epsilon\theta(s)$$

with $\psi(0) = 0$, then

$$\Delta x - \epsilon\psi(t) = \int_0^t \left[\frac{\partial f}{\partial x}(x^*, u^*)(\Delta x - \epsilon\psi(s)) + \frac{\partial^{\bar{2}} f}{\partial x^2)}\Delta x^2 + 2\frac{\partial^{\bar{2}} f}{\partial x \partial u}(\Delta x)(\epsilon\theta) + \frac{\partial^{\bar{2}} f}{\partial u^2}(\epsilon\theta)^2\right] ds.$$

Hence: $\Delta(x) = \epsilon\psi(t) + k_2(t)\epsilon^2$, where the function $k_2(t) \leq k_2$.

Now compute the variation in the cost integral due to the control variation $\theta(t)$.
$\Delta C = C(u^* + \epsilon\theta) - C(u^*) = \int_0^t \left[\frac{\partial f^0}{\partial x}(x^*, u^*)(x^*(s) - x(s)) + \frac{\partial f^0}{\partial u}(x^*, u^*)\epsilon\theta(s)\right] ds +$

$$\int_0^t \left[\frac{\partial^2 \bar{f}^0}{\partial x^2)}\Delta x^2 + 2\frac{\partial^2 \bar{f}^0}{\partial x \partial u}(\Delta x)(\epsilon\theta) + \frac{\partial^2 \bar{f}^0}{\partial u^2}(\epsilon\theta)^2\right] ds.$$

The first variation of $C(u^*)$ is (ΔC up to terms of order ϵ)

$$\delta C = \epsilon \int_0^t \left[\frac{\partial f^0}{\partial x}(x^*, u^*)\psi(s) + \frac{\partial f^0}{\partial u}(x^*, u^*)theta(s)\right] ds.$$

Use $\partial f^0/\partial x = \dot{\eta}^* + \eta^*(\partial f/\partial x)(x^+, u^*)$ and integrate by parts:

$$\delta C = \epsilon \int_0^T -\frac{\partial H}{\partial u}(\eta^*, x^*, u^*)\theta(s) ds = 0$$

The hypothesis (3) asserts that:

$$\frac{\partial^2 \bar{f}^0}{\partial x^2)}\Delta x^2 + 2\frac{\partial^2 \bar{f}^0}{\partial x \partial u}(\Delta x)(\epsilon\theta) + \frac{\partial^2 \bar{f}^0}{\partial u^2}(\epsilon\theta)^2 \geq \frac{c}{2}(|\Delta x|^2 + |\epsilon\theta|^2).$$

(α)) If $f_x^0 = 0$ on $(x^*(t), u^*(t))$ then

$$\Delta C \geq \frac{c}{2}\epsilon^2 \int_0^T |\theta(t)|^2 dt > 0$$

- In other case, there appears the extra complication in he second order terms of $\Delta x - \epsilon\psi$ arising in the simplifying of the expression for the first variation of δC.

(β)) In this case

$$\Delta C \geq T k_2 \epsilon^2 + \frac{c}{2}\epsilon^2 \int_0^T |\theta(t)|^2 dt > 0$$

11.4 Examples

In order to clarify the nature of the maximal principle let us consider the statement of the principle for autonomous linear control problem.

- $\dot{x} = Ax + Bu$ for real constant $n \times n$ matrix A and $n \times m$ matrix B
- Initial point x_0 and target $\mathcal{T}$ as the origin
- Compact convex restraint set $\Omega \subset R^n$
- $C(u) = \int_0^{t_1} dt = t_1$ the time duration of control.

In this case the Hamiltonian function is

$$\bar{H}(\bar{\eta}, \bar{x}, u) = \eta_0 + \eta[Ax + Bu] = \eta_0 + H(x; \eta, u)$$

where η is an n-row vector, and $\eta(Ax + Bu) = H$. Then:

$$\bar{M}(\bar{\eta}, \bar{x}) = \eta_0 + \eta Ax + \max_{u \in \Omega} \eta Bu = \eta_0 + M(\eta, x),$$

where $M = max_{u \in \Omega} H$.

If $u(t)$ on $0 \leq t \leq t_1$ is maximal, then the response $x(t)$ and adjoint response $\eta(t)$ satisfy:

$$\begin{aligned} \dot{x} =& \quad Ax + Bu \\ \dot{\eta} =& \quad -\eta A, \end{aligned}$$

and $x_0 = t, \ \eta_0 = constant$.

The maximal principle requires that:

$$\eta_0 + \eta(t)Ax(t) + \eta(t)Bu(t) = \eta_0 + \eta(t)Ax(t) + \max_{u \in \Omega} \eta(t)Bu(t)$$

or

$$\eta(t)Bu(t) = \max_{u \in \Omega} \eta(t)Bu(t)$$

almost everywhere on $0 \leq t \leq t_1$.

The second condition of the maximal principle asserts that:

$$\eta_0 + \eta(t)Ax(t) + \max_{u \in \Omega} \eta(t)Bu(t) = 0$$

everywhere.

If $\eta(t)$ vanished at one point on $0 \leq t \leq t_1$ then it vanishes identically since its a solution of the homogeneous linear system $\dot{\eta} = -\eta A$.

But if $n(t)$ vanished identically, then $\eta_0 = 0$ and this contradicts the non-vanishing of the $(n + 1)$ vector $\bar{\eta}$.

Therefore $\eta(t)$ is nowhere zero.

In this case, we can ignore the response components x_0 and $\eta_0 = constant$, and find the maximal controller in terms of $H(\eta, x, u)$ and $M(\eta, x)$.

It is important to notice that the adjoint response $\eta(t)$ satisfies the fixed differential system $\dot{\eta} = -\eta A$ whose coefficient does not depend on the control $u(t)$ or the response $x(t)$. Thus $\eta(t)$ is entirely determined by its initial conditions.

Consider the important case in where Ω is the m-cube $|u^j| \leq 1$.

Then

$$\eta(t)Bu(t) = \max_{u\in\Omega} \eta(t)Bu(t)$$

means that each component of the maximal controller $u^i(t)$ should be chosen as $+1$ or -1 according to the sign of the corresponding component of the vector $\eta(t)B$. That is the maximal controller must satisfy

$$u(t) = [sign(\eta(t)B)]^T.$$

almost everywhere.

Example 70 *Let* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- $\dot{\mathbf{x}} = Ax + Bu \rightarrow \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{cases}$
- $\dot{\eta} = -\eta A \rightarrow \begin{cases} \dot{\eta}_1 &= 0 \\ \dot{\eta}_2 &= -\eta_1 \end{cases}$
- $u(t) = sgn(\eta(t)B) = sgn(\eta_2(t)).$

The *optimal response from* x_0 *to the origin must describe an arc of solution of the extreme differential system:*

- $u = 1 \rightarrow \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -1 \end{cases}$ [**A**]
- $u = -1 \rightarrow \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 1 \end{cases}$ [**B**]

Since the extreme differential systems [**A**] and [**B**] are autonomous we can construct extreme responses ending at the origin, using a process that shifts as t increases.

12 Controllability of non linear processes

Theorem 71 *Consider the control process in* R^n

$$\dot{x} = f(x,u) \;\; in \; C^1 \;\; in R^{n+m},$$

with $u = 0$ *in the interior of the restraint set* $\Omega \subset R^m$.

Assume that:

- *(a)* $f(0,0) = 0$;

- *(b)* $rank[B, AB, ..., A^{n-1}B] = n$, *where* $A = f_x(0,0),\ B = f_u(0,0)$.

Then the domain of controllability $\mathcal{C} = R^n$.

Proof: (Idea) Consider the time reversed differential system:

$$\dot{x} = -f(x, u),$$

and then it is possible to prove that the set of endpoints $x(1)$ of responses of the time reversed system initiating at $x(0) = 0$, covers an open neighborhood N about $x = 0$. Then, by reversing the time sense again on each appropriate controller, we note that each point in N can be steered to the origin along a solution of the original system. The proof is giving in [Lee, E.; Markus, L.] pag. 366.

13 Some applications to economics: Intertemporal optimization

The tools of calculus of variations and optimal control theory have been used to analyze many dynamic questions in economics. An early application of these tools to the topic of optimal economic growth was due to Ramsey (1928). Then the problem was almost forgotten for some time probably as a result of the Great Depression and the war. Then in 1950s there was a revival of the problem, (Koopmans and Cass). In the early 1960s Cass formulated the problem in terms of Pontriaguim' s maximun principle. Solow have summarized more recently the work using modern control theory.

Mathematically, optimal control theory is closely related with the calculus of variations, optimal control theory incorporates general constraints imposed on the problem in a direct and natural way. Pontriaguin and his associates are chiefly responsible for this approach.

13.1 An illustrative example:

The optimal growth problem.

- The capital per unit of labor $k(t) = K(t)/L(t)$.
- $\dot{k}(t)$ represents the rate of change in capital per unit of labor.
- The initial value of the capital $K(0) = K_0$.
- The consumption per unit of labor $c(t) = C(t)/L(t)$
- Labor growths at the constant proportional rate n. So $L(t) = L_0 e^{nt}$.

- The function of production is of the form $Y(t) = F(K(t), L(t))$ homogeneous of degree one.
- So, with the usual notation $q(t) = f(k(t))$.
- We suppose a depreciation proportional to the capital stock $\theta K(t)$.
- The condition of equilibrium is $Y(t) = C(t) + I(t)$ then it follows:

$$\frac{dK(t)}{dt} = \dot{K}(t) = F(K(t), L(t)) - C(t) - \theta K(t).$$

  As, $k(t) = K(t)/L(t)$ substituting in the equilibrium condition, we obtain

$$\dot{k} = f(k(t)) - c(t) - \lambda k(t)$$

  where $\lambda = \theta + n$.

- It is assumed that:

$$f'(k(t)) > 0, \quad f''(k(t)) < 0, \quad f(0) = 0, \quad f(\infty) = \infty,$$

$$\lim_{k \to 0} f'(k(t)) = \infty, \quad \lim_{k \to \infty} f'(k(t)) = 0.$$

The general optimal control problem with finite horizon is given by: Maximize the welfare function

$$\max_{c} \int_0^T u(c(t)) e^{-\delta t} dt$$

subject to:

$$\begin{aligned} &\dot{k} = f(k(t)) - c(t) - \lambda k(t) && (\mathbf{a}) \\ &0 \leq c(t) \leq f(k(t)) && (\mathbf{b}) \\ &k(0) = k_0, \ \ k(T) = k_T \ \ k(t) \geq 0, \ \ c(t) \geq 0, && (\mathbf{c}) \end{aligned} \tag{24}$$

where δ is a discount rate, and utility per unit of consumption is given by $u(c(t))$ $u'(c) > 0$, $u''(c) < 0$, represent positive but diminishing marginal utility (concavity of the utility function).

Observe that in the above problem there appears joint with the initial condition (24 **c**), a new equation (24 **b**)that restraint the set of possible solutions. The control variable C is confined to every point of time to the control region $[0, f(k)]$, which is just another way of saying that the marginal propensity tu consume is restricted to the interval $[0, 1]$.

By coincidence, it may happen that when we ignore this type of constraints and solve the given problem as an unconstrained one, the optimal path lies entirely in the permissible area. In this event, the constraint is trivial. But we would expect the unconstrained path optimal to violate the constraint.We shall consider this situation only further on when we consider the Hesteness's theorem.

We can consider this problem as a Control Optimal Problem; whose Hamiltonian function is defined by:

$$\bar{H}(k, c, t, \eta) = \eta_0 u(c(t))e^{-\delta t} + \eta(t)\left[f(k(t)) - c(t) - \lambda k(t)\right]$$

The maximal principle implies

$$\eta_0 u(c(t))e^{-\delta t} + \eta(t)\left[f(\bar{k}(t)) - c(t) - \lambda \bar{k}(t)\right]$$
$$\leq \eta_0 u(\bar{c}(t))e^{-\delta t} + \eta(t)\left[f(\bar{k}(t)) - \bar{c}(t) - \lambda \bar{k}(t)\right] \tag{25}$$
$$\text{for all } \; c(t) \geq 0, \;\; 0 \leq t \leq T.$$

Now if $\eta_0 = 0$ (recall that η_0 is a constant) then $\eta(t)c(t) \geq \eta(t)\bar{c}(t)$ for all $c(t) \geq 0, \;\; 0 \leq t \leq T$. Since η_0 and $\eta(t)$ cannot vanish simultaneously, we obtain $\eta(t) \not\equiv 0$.

- If $\eta(t) > 0$, it follows that $\bar{c}(t) \leq c(t)$ for all admissible $c(t)$. Set $c(t) \equiv 0$ then it follows that $\bar{c}(t) \equiv 0$.
- If we impose the Koopmans condition $\lim_{x\to 0, x>0} u(x) = -\infty$ then $\bar{c}(t) = 0 \;\; \forall t$ cannot be optimal. This assumptions *means the existence of a strong incentive to avoid periods of very low consumption as much as possible.* If $c(t) = 0$ for any small time interval, the objective integral diverges to $-\infty$. In essence this condition guarantees an interior solution.

  However we don't need this condition to guarantee that $c(t) \equiv 0$ is not optimal. To see this notice that from the conditions imposed to the production function it is possible to obtain $c(t) \geq 0$ and this path is better than $\bar{c}(t) \equiv 0$.
- Alternatively we can justified this by means of the Inada condition: $\lim_{x\to 0, x>0} u'(x) = \infty$
- If $\eta(t) < 0$, it follows that $\bar{c}(t) \geq c(t)$; for all $c(t) \geq 0$, thus we must consider as possible optimal solution $\bar{c}(t) = f(\bar{k}(t))$ where $\bar{k}(t)) = k_0 e^{-\lambda t}$. If at $t = T$ the equality $k_T =_0 e{-}\lambda T$ follows then $(\bar{c}, \bar{k})$ is the optimal solution.

So, a solution obtained from $\eta = 0$ exists in very particular cases. By another part recall that we are using a necessary condition for the an optimal controller, but non-sufficient. Observe that this solution can be discarded if we do not impose the restraint (24 b). This kind of solutions suppose that there is not investment in capital in the predicted term of the process.

Let us now consider the case $\eta_0 > 0$. We can choose $\eta_0 = 1$ without loss of generality. The inequality(25) can be rewritten as:

$$[u(\bar{c}(t)) - u(c(t))]\, e^{-\delta t} \geq \eta(t)\bar{c}(t) - \eta(t)c(t) \tag{26}$$

this inequality and the Koopman's condition implies that $\bar{c}(t) \geq 0, \quad \forall t, 0 \leq t \leq T$. So that we have an interior solution.

The inequality (26) means that the consumer maximize his satisfaction from the consumption along the optimal path for each instant of time, over those consumptions whose values do not exceed the value at implicit prices $\eta(t)$ of the optimal consumption $\bar{c}(t)$ because we obtain that

$$[u(\bar{c}(t)) - u(c(t))] \geq 0$$

for all $c(t) \geq 0,\ 0 \leq t \leq T$, such that $\eta(t)\, [\bar{c}(t) - c(t)] \geq 0$

And since

$$[u(\bar{c}(t)) - u(c(t))] \leq 0$$

for all $c(t) \geq 0,\ 0 \leq t \leq T$, then $\eta(t)\, [\bar{c}(t) - c(t)] \leq 0$. Then it follows that the consumer minimizes at each instant his expenditure over those consumption paths which would give him satisfaction, that is higher or equal than the satisfaction obtained form $c\bar{(t)}$.

In the above interpretations, it is clear that the Pontriaguin auxiliary variable, $\eta(t)$ plays the role of the implicit (or shadow) price.

The hamiltonian is seem to contain a hump, with its peak occurring at a c between $c = 0$ and $c = c_1$, where c_1 is the solution of the equation $f(k(t)) - c(t) - \lambda k(t) = 0$ it follows that $c_1 < f(k)$. Hence the maximun corresponds to a value of c in the interior of the control region $[0, f(k)]$. We can acordingly find the maximun of H by setting

$$\frac{\partial H}{\partial c} = u'(c)e^{-\delta t} - \eta(t) = 0. \tag{27}$$

From this we obtain the condition $u'(c) = \eta e^{\delta t}$ which states that the optimal marginal utility of per-capita consumption should be equal to the shadow price of capital amplified by the exponencial $e^{\delta t}$. The condition on the utility function guarantee the fact tha in this point H is indeed maximized.

The Hamiltonian system for the present problem is:

$$\begin{aligned} \dot{k} = &\quad f(k(t)) - (\theta + n)k(t) - c(t) \left(= \tfrac{\partial \bar{H}}{\partial \eta}\right) \\ \dot{\eta} = &\quad -\eta\, [f'(k(t)) - \lambda] \left(= \tfrac{\partial \bar{H}}{\partial k}\right) \end{aligned}$$

And from the maximal principle: $\frac{\partial \bar{H}}{\partial c} = 0$ it follows:

$$\eta(t) = u'[c(t)]e^{-\delta t}. \tag{28}$$

Since $u'[x(t)] > 0$ we obtain:

$$\dot{c} = -\frac{u''(c)}{u'(c)}\left[\frac{\partial f}{\partial k}(k(t)) - (\theta + n + \delta)\right]$$

Define

$$P(t) = \eta\left[f'(k(t)) - \lambda\right].$$

That is $P(t)$ is the present value of the net marginal productivity at time t it verify $\dot{\eta} = -P(t)$.

13.2 Transversality conditions

Now suppose that we alter the above problem in such a way that the end state $k(T)$ is *not a priori* specified, but T is fixed. With this modification, the above analysis hold as it is, except for two crucial points:

1. We do not have to prove that $\eta_0 > 0$ since $\eta_0 = 1$ for all t,and
2. the transversality condition on $\eta(T) = 0$ (see theorem (69)) implies $u'(c(T))e^{-\delta T} = 0$. As long as T is finite the condition $u'(c) > 0$ (no satiation on consumption) should be modified. See theorems (54) and (66).

However following, [Arrow, K.; Hurwitz, L.; Uzawa, H.] we can modify the objective functional as follows:

$$\int_0^T u(c(t))e^{-\delta t}dt + k(T)$$

and we maintain the no satiation condition.

The transversality condition in this case should be rewritten as:

$$\eta(T) \geq 0, \; and \; \eta(T)k(T) = 0,$$

see section (13.2). So, $u'(c(T))e^{-\rho t} > 0 \;\; and \;\; u'(x(T))k(T)e^{-\rho t} = 0$.

Hence if we have $u'(c(T)) > 0$ for all x so that $\eta(T) > 0$ we must have $k(T) = 0$, this means that is always better to *eat up* the capital to increase consumption for some period of time and leave nothing after the planning horizon. Thus $k(0) = k_0$ and $k(T) = 0$ specifies the two boundary conditions for the Hamiltonian system.

13.3 The Model with infinite horizon

Subsection (9.5) is the reference for the following application.

Maximize the welfare function

$$\max_c \int_0^\infty u(c(t))e^{-\delta t}dt$$

subject to:

$$\begin{aligned} &\dot{k} = f(k(t)) - c(t) - \lambda k(t) && (\mathbf{a}) \\ &0 \le c(t) \le f(k(t)) && (\mathbf{b}) \\ &k(0) = k_0, \;\; k(t) \ge 0, \;\; c(t) \ge 0, && (\mathbf{c}) \end{aligned} \tag{29}$$

where δ is a discount rate, and utility per unit of consumption is given by $u(c(t))$ $u'(c(t)) > 0$, $u''(c(t)) < 0$, represent positive but diminishing marginal utility (concavity of the utility function).

As long as we do not specify the terminal stock $k(T)$ when $t \to \infty$, the problem is identical with the usual growth problem, except in one important aspect: *How should the transversality condition be modified for the infinite horizon problem ?* Observe that when $T \to \infty$ the problem of non satiation discussed above does not arise, since $\eta(T) = 0$ when $T \to \infty$.

Although the condition $\eta(T) \to 0$ *as* $T \to \infty$ may appear to be a natural extension of the transversality condition $\eta(T) = 0$ for finite T, counterexamples can be shown where this is not true. However this condition is necessary and sufficient for linear systems, see section(7.6)

The real question here is *whether such a condition indeed constitutes a condition for optimality.* In general, appropriate conditions for the infinite horizon problem, which replace the transversality conditions for the finite horizon problem, are not known.

The question we have to ask is **What is a transversality condition at infinite ?**

In [Arrow, K.; Hurwitz, L.; Uzawa, H.] is pointed out the following condition, as long as $\delta > 0$:

$$\lim_{T\to\infty} \eta(T) \ge 0, \text{ and } \lim_{T\to\infty} \eta(T)k(T) = 0.$$

However this condition is false when $\delta = 0$ The condition $\lim_{T\to\infty} u'(c(T))k(T) = 0$ does not hold in general. When $\delta = 0$ the following condition is necessary:

$$\lim_{T\to\infty} u'(c(T)) = u'(c_1) \quad and \quad \lim_{T\to\infty} k(T) = k_1, \tag{30}$$

or

$$\lim_{T\to\infty} \eta(T) = u(c_1) \quad and \quad \lim_{T\to\infty} \eta(T)k(T) = u'(c_1)k_1. \tag{31}$$

Observe that condition 31 is a counterexample to the conjecture that the transversality condition is simply extended to the infinite horizon problem by setting $T \to \infty$

13.4 The steady state

In a steady state we should have $\dot{k} = 0$. It follows that the capital per unit of labor will be constant if $\dot{k} = 0$, this will be truth on a curve $c_t(k) = f_t(k) + (\theta + n)k_t$, so that $c_t(k) \leq f_t(k)$.

This curve has the following characteristics:

- It pass by the origin, $k = c = 0$, with infinite derivative, because $lim_{k \to 0} f'(k(t)) = \infty$
- It has a maximun at k^* where $\frac{\partial f}{\partial k}(k^*) - \theta - n = 0$, that corresponds to the *Golden Ruler.* At this point the marginal productivity of capital equalizes the rate of the population growth and a maximization of the consumption per unit of labor is reached.

The condition of the stationary consumption per unite of labor $\dot{c}(t) = 0$, is reached at the point $\bar{k}$ where

$$\frac{\partial f}{\partial k}(\bar{k}) = (\theta + n + \delta),$$

this point corresponds to *the modified Golden Rule* .

- Consider the plans whose axes are k and c; the condition of the stationary evolution is satisfied in the intersection of the vertical straightline trough $\bar{k}$ and the curve given by $c_t(k) = f_t(k) - (\theta + n)k_t$.
- For each k_0 there exists a unique initial level c_0 on the consumption, and an unique trajectory that starting at this initial point, reaches the stationary point in an infinite time.
- A point like this is called a saddle-point, and the path is called the **modified golden rule path.**

Note that since $\eta(t) = u'[c(t)]e^{-\delta t}$ (the maximal principle) and specially $u''(c) < 0$ it follows that the demand function is

$$\bar{c}(t) = g[p(t)]$$

where:

$$p(t) = \eta(t)e^{\delta t} = u'(\bar{c}(t)), \;\; g \equiv (u')^{-1}, g' < 0, \text{ for all } \; p.$$

The condition $\lim_{x \to 0, x>0} u'(x) = \infty$ allows us to assert that the solution is an interior point, that is: $\bar{c}(t) > 0$ for all t.

We can rewrite the Hamiltonian system as

$$\dot{\bar{k}}(t) = f(\bar{k}(t)) - \lambda \bar{k}(t) - g(p(t)) \tag{32}$$

$$\dot{p} = -p(t)[f'(\bar{K}(t)) - (\lambda + \delta)] \tag{33}$$

13.5 The saddle path

For any positive initial level of k there is a unique initial level of c that is consistent with households' intertemporal optimization. The function giving this initial c as a function of k is know as the saddle path. For any starting values of k, the initial c must be the value on the saddle path. The economy the moves along this saddle path to point. E..

A natural question is whether the equilibrium of this economy represents a desirable outcome. The household's optimization problem requires that paths where capital stock becomes negative be ruled out, and also path that cause consumption to approach zero must be ruled out because they do not maximize household's utility. In short the solution is for the initial value of c to be given by the value on the saddle path, and for c and k moving along the saddle path.

Once the economy has converged to the saddle point E capital, output, and consumption per unit of labor are constant. Since y and c are constant, the saving rate $(y - c)/y$, is also constant. The capital stock, total output and consumption growth at rate n. The effectiveness of labor remains the only possible source of persistent growth in output per worker. We shall consider this point further on. See section (13.8).

13.6 Halkin's counterexample for the condition: $\lim_{T\to\infty} \eta(T) = 0$.

In view of the importance of the problem, we show in this section that the condition that $\eta(T) = 0$ as $T \to \infty$ may fail to hold for the infinite horizon problem.

Consider a control problem which maximizes:

$$\int_0^\infty [1 - y(t)]v(t)dt$$

subject to $\dot{y}(t) = [1 - y(t)]v(t), \ -1 \leq v(t) \leq 1$, and $y(0) = 0$.

Observe that

$$\int_0^\infty [1 - y(t)]v(t)dt = \int_0^\infty \dot{y}(t)dt = \lim_{t\to\infty} y(t).$$

By direct integration $y(t) = 1 - e^{-V(t)}$ where $V(t) = \int_0^T v(\tau)d\tau$. Hence $y(t) < 1, \ \forall t$. Hence any election of $v, -1 \leq v \leq 1$ for which $\lim_{t\to\infty} V(t) = \infty$ is optimal. For example $v(t) = v_0(constat)$ is optimal.

The Hamiltonian for this problem is:

$$H = [1 + \eta(t)][1 - y(t)]v(t).$$

The maximal principle implies $\partial H/\partial v = [1 + \eta(t)][1 - y(t)] = 0$ Hence $\eta(t) = -1$ for all t since $y(t) < 1$ for all t. Owing the continuity of $\eta(t), \lim_{t\to\infty} \eta(t) = -1$ and **not** 0.

13.7 Optimal Growth with a linear objective function

Consider the objective integral defined as:

$$J = \int_0^\infty c(t)e^{-\rho t}dt, \text{ where } \rho > 0.$$

The constraints, are: $\dot{k} = f(k(t)) - \lambda k(t) - c(t), \ \ and \ \ c(t) \geq 0, k(t) \geq 0.$

Let $s(t)$ be the propensity to save at time t :

$$s(t) = \frac{y(t) - c(t)}{y(t)} = \frac{f(k(t)) - c(t)}{f(k(t))}$$

We rewrite the objective integral and the constraint as follows:

$$J = \int_0^\infty (1 - s(t))f(k(t))e^{-\rho t}dt,$$

$$\dot{k} = s(t)f(k(t)) - \lambda k(t); \ 0 \leq s(t) \leq 1, k(t) \geq 1.$$

In this way the control variable is s and the state variable k. The problem is to choose the time path of $s(t)$ so as to maximize J with a given k_0.

The Hamiltonian for this problem is:

$$H[k(t), s(t), t, \eta(t)] = e^{-\rho t}(1 - s(t))f(k(t)) + \eta(t)[s(t)f(k(t)) - \lambda k(t)]$$

The hamiltonian system consists of the following equations:

$$\dot{k} = s(t)f(k(t)) - \lambda k(t)$$

$$\dot{\eta} = -e^{-\rho t}(1 - s(t))f''(k(t)) + \eta(t)[s(t)f(k(t)) - \lambda)]$$

and the transversality condition $lim_{t\to\infty}\eta(t) = 0$.

Observe that the Hamiltonian is a linear equation in the control. Following the Pontriaguin maximal principle we obtain a corner solution given by:

$$s(t) = 1 \quad if \quad -e^{-\rho t} + \eta(t) > 0$$

$$s(t) = 0 \quad if \quad -e^{-\rho t} + \eta(t) < 0$$

Define $p(t) = \eta(t)e^{\rho t}$ then:

$$s(t) = 1 \quad if \quad p(t) > 1$$

$$s(t) = 0 \quad if \quad p(t) < 1$$

Then $\dot{\eta}(t) = e^{-\rho t}\dot{p}(t) - \rho e^{-\rho t}p(t)$. Then from the adjoint equation we obtain:

$$\dot{p}(t) = (\lambda + \rho)p(t) - \pi(t)f(k(t))$$

where $\pi(t) = (1 - s(t)) + s(t)p(t)$.

In terms of $p(t)$ the transversality condition can be rewritten as

$$\lim_{t\to\infty} p(t)e^{-\rho t} = 0$$

And the Hamiltonian can be rewritten as: $H = e^{-\rho t}[\pi(t)f(k(t)) - \lambda k(t)p(t)]$.

- (1) **Case** $s(t) = 1$. The Hamiltonian system in this case is:

$$\dot{k} = f(k) - \lambda k$$

$$\dot{p} = -p[f'(k) - \lambda - \rho]$$

Let $\bar{k}$ and k^* be respectively defined by the following equations:

$$f(\bar{k}) = \lambda\bar{k}$$

$$f'(k^*) = \lambda + \rho$$

Assuming that the initial capital k_0 is less than k^*, there exists a path $(k(t), p(t))$ which starts in $[k_0, p(k_0)]$ and reach in a finite time T the state: $[k^*, 1]$. Define now the path $(k^1(t), q^1(t))$ which is the same as the above path for the period $0 \leq t \leq T$, but is $(k^*, 1)$ for $t > T$.

We may now examine when the path $(k^*, 1)$ satisfies the Hamiltonian system.

- First note that along this path $\dot{k} = \dot{p} = 0$ then the Hamiltonian system is reduced to:

$$0 = s_t f(k^*) - \lambda k^* \ \ and \ \ 0 = (\lambda + \rho) - f'(k^*)$$

- The second equation is obviously satisfied by the definition of k^*. The first equation is verified only if s_t takes the value:

$$s_t = s^* = \frac{\lambda k^*}{f(k^*)} \ \ for\ all \ \ t.$$

- Note that $s^* > 0$ and that $k^* < \bar{k}$ implies $s^* < 1$. Hence $(k^*, 1)$ verifies the restrictions and the equations of the problem.

- Then the path $(k^1(t), p^1(t))$ verifies all the conditions of the Pontriaguin Maximun Principle.

- (2) **Case** $s(t) = 0$. The Hamiltonian system in this case is:

$$\dot{k} = \lambda k$$

$$\dot{p} = p(\lambda + \rho) - f'(k)$$

If $k_0 > k^*$ There exists a path $(k(t), p(t))$ starting at $[k_0, q(k_0)]$ and reaching in a finite time T' the state: $[k^*, 1]$. Define now the path $(k^2(t), q^2(t))$ which is the same as the above path for the period $0 \leq t \leq T'$, but is $(k^*, 1)$ for $t > T'$. It is easy to see that this path verifies all three conditions of the maximun principle including the transversal condition: $lim_{t\to\infty} p(t)e^{-\rho t} = 0$.

Theorem 72 *For the above model, given an arbitrary initial value of* k*, there is a unique optimal attainable path which is characterized as follows:*

1. $k_0 < k^*$, $s(t) = 1$, *and after* $k(t)$ *reaches* k^*, $k(t) = k^*$ *for all such* t.

2. $k_0 > k^*$; $s(t) = 0$,*and after* $k(t)$ *reaches* k^*, $k(t) = k^*$ *for all such* t.

3. $k_0 = k^*$: $k(t) = k^*$ *for all* t *and* $s(t) = s^* = \frac{\lambda k^*}{f(k^*)}$.

In other words, the optimal attainable path is the one that reaches the modified *golden rule path* where $c^* = (1 - s^*)f(k^*)$, with a maximun speed and stays thereafter on it.

Thus the solution path is such that if $k_0 < k^*$, the economy maximizes saving from current income until time T, and after time T maintain a constant saving ratio s^*. Analogously in the case $k_0 > k^*$ the economies minimize saving from current income until time T' and after time T' maintain a constant saving ratio s^*.

Remark 73 *It may be noted that even at the stationary point, the per-capita consumption becomes constant, and its level cannot increased further over time.*

- *This is because a static production function* $Y = Y(K, L)$ *is assumed on the model.*

- To make possible a rising per-capita consumption, technological progress must be introduced.

13.8 Exogenous and endogenous technical progress

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It is plausible that technological progress is the reason that more output can be produced today from a given quantity of capital and labour than a century or two ago. To see this we introduce an explicit *research and development* or (R&D) sector. Following [Romer, D.] we consider a model with four variables: labor (L), capital (K), technology (A) and output (Y). There are two sectors, a goods production sector where output is produced and an R&D sector where additions to the stock of knowledge are made.

The quantity of output produced at time t is: $Y = F(K, L(t)),\ \ where\ \ \dot{Y} = dY/dt > 0$,

- the positive sign of the derivative, shows that technological progress take place, but it offers no explanation of how this progress comes in to being.

- An alternative is to introduce a technological variable explicitly.

$$Y = F(K, A(t)L(t)), \eta = AL$$

  is referred to as *effective labor* where $A(t) > 0,\ and\ Y_A > 0$. So, we can write $Y = F(k, \eta)$, and $k_\eta = \frac{K}{\eta},\ \ y_\eta = \frac{Y}{\eta}\ \ and\ \ y_\eta = \phi(k_{eta})$. Now if we consider η rather than L as the relevant labor input variable, then $Y = F(K, \eta)$ can be treated mathematically just as a static production function.

The optimal control problem is formally the same than in the static case, the steady state in which Y, K and η all grow at the same rate is the intersection of $\dot{c}_\eta = 0$ and $\dot{k}_\eta = 0$.

- In the static model we had: $c(t) = C/L$ constant in the steady state.

- Now we obtain: $c_\eta(t) = \frac{C}{\eta}$ which implies that $\frac{C}{L} = c_\eta(t)A$

  Thus as A increases as a result of technological progress, per-capita consumption C/L will rise over time.

To understand the role that the technical progress and the scientific knowledge play in growth theory, we shall consider two simplified models:

1. An exogenous growth model: following the spirit of the Solow model, where the saving rates are taken as given, and

2. an endogenous growth model: where the saving behavior is modeld arising from the choices of optimizing individuals.

13.8.1 Exogenous growth model

This model has nothing to do with optimal control theory however, we consider that it plays an important role to understand the value of the scientific knowledge and technology in the growth over time in the standards of living. And on the other hand, it is a good introduction to the next model that is straightforward application of the optimal control theory.

We make two other major simplifcations. First, both the R&D and good production functions are assumed to be generalized Cobb-Douglas functions. Second in the spirit of the Solow model, the model takes the fraction of output saved s and the fractions a_L of the labor force and the fraction of the capital stock used in the R&D are exogenous and constant.

The quantity of output produced at time t is thus:

$$Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)]^{1-\alpha}, \quad 0 < \alpha < 1. \tag{34}$$

The production of news ideas depends on the quantities of capital and labor engaged in research and on level of technology:

$$\dot{A}(t) = G(a_K K(t), a_L L(T), A(t)). \tag{35}$$

Under the assumption of generalized Cobb-Douglas production this becomes:

$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta}, \quad B > 0, \beta \geq 0, \gamma \geq 0. \tag{36}$$

We denote $g_A \equiv \frac{\dot{A}(t)}{A(t)}$.

The dynamics of knowledge and capital. Following our assumptions the expression for capital accumulation yields:

$$\dot{K}(t) = s(1 - a_K)^{\alpha}(1 - a_L)^{1-\alpha} K(t)^{\alpha} A(t)^{1-\alpha} L(t)^{1-\alpha}.$$

Dividing both sides by $K(t)$ and defining $c_K = s(1 - a_K)^{\alpha}(1 - a_L)^{1-\alpha}$ gives us:

$$g_K(t) \equiv \frac{\dot{K}(t)}{K(t)} = c_K \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}.$$

Thus whether g_k is rising, falling, or holding steady depends on the behavior of AK/L. The growth rate of this ratio is $g_A + n - g_K$. Thus g_K is rising if this ratio is positive, falling if it is negative and constant if it is zero.

The major forces governing the allocation of resources to the development of knowledge are:

1. Support for scientific research.

2. Private incentives for R&D and innovation.

3. Alternative opportunities for talented individuals.

4. Learning by doing.

- **Support for scientific research** Basic scientific knowledge has traditionally been made available relatively freely, the same is true of the results of research undertaken in such institution as modern universities and medieval monasteries. Thus this research is not motivated by the desire to earn private returns in the market. Instead it is supported by governments, charities, and wealthy individuals and is pursued by individuals motivated by this support, by desire for fame, and perhaps even by love of knowledge. Since it is given away at zero cost and since it is useful in production, it has a positive externality. Thus its production should be subsidized.

- **Private incentives for R&D and innovation.** Many innovations or small improvements in existing goods receive little or no external support and are motivated almost entirely by the desire of private gain. The modelling of these private R&D activities and their implications for economic growth has been the subject of considerable recent research. See for instance [Romer, P. M. 1997].

- **Alternative opportunities for talented individuals.**

  Several authors observe that major innovations and advances in knowledge are often the result of the work of extremely talented individuals. These observations suggest that the economic incentives and social forces influencing the activity of highly talented individuals may be important to the accumulation of knowledge.

- **Learning by doing.** In this point, the central idea is that as individuals produce goods, the inevitably think of ways of improving the production process. For example see [Arrow, K.].

13.8.2 Endogenous technological progress

The analysis in the previous section, following the spirit of the Solow model, takes the saving rate as given. But we sometimes want to model saving behavior as arising from the choices of optimizing individuals, particularly if we are interested in welfare. This model has two state variables, A an K and to control variables (α, and s). Obtain an explicit solution is not simple.

In this model the accumulation of knowledge explicitly depends on what amounts of resources are devoted to inventive activity. If $A(t)$ denotes the stock of knowledge:

$$\dot{A}(t) = \sigma\alpha(t)Y(t) - \beta A(t) \quad (0 < \sigma \leq 1,\ 0 \leq \alpha \leq 1, \beta \geq 0) \tag{37}$$

- where σ is the research success coefficient, $\alpha(t)$ denotes the fraction of output channelled toward inventive activity at time t, and β is the rate of decay of technical knowledge.

Out of remaining resources, a part will be saved (and invested); then the variable $K(t)$ changes over time according to:

$$\dot{K}(t) = s(t)[1 - \alpha(t)]Y(t) - \delta K(t),$$

where s denotes the propensity to save and δ denotes the rate of depreciation.

If the government seeks to maximize social utility then there arises the optimal control problem:

$$\begin{array}{ll} \max_{\alpha,s} & \int_0^\infty U[(1-s)(1-\alpha)Y]e^{-\rho t}dt \\ \textit{subject to} & \dot{A}(t) = \sigma\alpha(t)Y(t) - \beta A(t) \\ & \dot{K}(t) = s(t)[1-\alpha(t)]Y(t) - \delta K(t) \\ \textit{and} & A(0) = A_0, \quad K(0) = K_0. \end{array} \tag{38}$$

Observe that the objective looks unfamiliar, but it is another way to express $U(C)e^{-\rho t}$, because $C = Y - \alpha Y - s(1-\alpha)Y = (1-s)(1-\alpha)Y$.

Example 74 *Consider the following control problem with two state variables, A and K with and two control variables, C and S_A. We consider S as skill or skilled labor (human capital). S can be used for the production of the final good, Y, or for improvement of technology A. We have: $S = S_Y + S_A$. Technology A, is not fixed. It can be create by engaging human capital S_A as follows:*

$$\dot{A} = \sigma S_A A,$$

where σ is the research success parameter. The production function is assumed to be of the Cobb-Douglas type:

$$Y = (S_Y A)^\alpha (AL_0)^\beta K^{1-\alpha-\beta}.$$

We assume that L is constant.

Consider the specific case were

$$U(C) = \frac{C(t)^{1-\theta}}{1-\theta} \quad \theta > 0.$$

The control problem takes the form:

$$\max_{C,S_A} \int_0^{\infty} \frac{C(t)^{1\theta}}{1-\theta} e^{-\rho t} dt$$

$$subject\ \ to\ \ \ \dot{A} = \sigma S_A A$$

$$\dot{K} = (S_0 - S_y)^{\alpha} A^{\alpha+\beta} L_0^{\beta} K^{1-\alpha-\beta} - C$$

$$and\ \ \ A(0) = A_0,\ K(0) = K_0.$$

Then we have the current value Hamiltonian:

$$H_c = \frac{C(t)^{1\theta}}{1-\theta} + \lambda_A(\sigma S_A A) + \lambda_K(\Delta - C),$$

where $\Delta = (S_0 - S_y)^{\alpha} A^{\alpha+\beta} L_0)^{\beta} K^{1-\alpha-\beta}$, $H_c = He^{\rho t}$, and $\lambda_A = \eta_A e^{\rho t}, \lambda_K = \eta_K e^{\rho t}$, are the shadow prices of A and K.

We get the conditions:

$$\begin{aligned} &\tfrac{\partial H_c}{\partial C} = C^{-\theta} - \lambda_K \ \Rightarrow\ C^{-\theta} = \lambda_K \\ &\tfrac{\partial H_c}{\partial S_A} = \lambda_A \sigma A - \lambda_K \alpha (S_0 - S_y)^{-1} \Delta = 0 \end{aligned} \tag{39}$$

$$\Rightarrow \Delta = \frac{\lambda_A \sigma A}{\lambda_K \alpha}(S_0 - S_y).$$

In addition to the $\dot{A}$ and $\dot{K}$ given in the problem statement, the P.M.P requires that :

$$\begin{aligned} &\dot{\lambda}_A = \tfrac{\partial H_c}{\partial A} + \rho\lambda_A = -\lambda_A \sigma S_A - \lambda_K \alpha (S_0 - S_y) A^{-1} \Delta + \rho \lambda_A \\ &\dot{\lambda}_K = \tfrac{\partial H_c}{\partial K} + \rho\lambda_k = -\lambda_K (1 - \alpha - \beta) K^{-1} \Delta + \rho \lambda_K. \end{aligned} \tag{40}$$

The steady state

The basic feature is that the variables Y, K, A, and C al grow at the same rate:

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{A}}{A} = \frac{\dot{C}}{C} = \sigma S_A.$$

We can calculate:

$$\frac{\dot{\lambda}_\theta}{\lambda_\theta} = -\theta \frac{\dot{C}}{C} = -\theta \sigma S_A.$$

and

$$\frac{\dot{\lambda}_A}{\lambda_A} = \rho - \sigma \left(\frac{(\alpha+\beta)}{\alpha} S_0 - \frac{\beta}{\alpha} S_A \right)$$

Then it follows that in steady state S_A is constant

$$S_A = \frac{\sigma(\alpha + \beta)S_0 - \alpha\rho}{\sigma(\alpha\theta + \beta)}$$

Parametrically expressed steady state growth rate is

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{A}}{A} = \frac{\dot{C}}{C} = \frac{\sigma(\alpha + \beta)S_0 - \alpha\rho}{\sigma(\alpha\theta + \beta)}.$$

Visual inspection is sufficient to establish that human capital $S_=$ has a positive effect on the growth rate, as does the research success parameter σ. But a negative effect is exerted by the discount rate ρ. More formally the effects of the various parameters can be found taking the partial derivatives in this expression.

13.9 The discounting rate and bifurcations

Consider the general multi-sector optimal growth problem formulated by [Benhabib, J.; Nishimura, K.]

$$\max_y \int_0^\infty e^{-(\delta-n)t} U(c(y,k))dt$$

$$s.t.\ \ \dot{k}_i = y_i - nk_i, \quad i = 1, ..., n,$$

where y is the vector per-capita outputs y_i in sector i k as the vector of per-capita stock of capital, c is the consumption, U is the utility from consumption, δ is the discounting rate and n as the population growth rate.

The Hamiltonian function of this problem is:

$$H(y, k, \eta) = e^{-(\delta-n)t} U(c(y,k)) + \eta(y - nk).$$

Then the current-value hamiltonian, denoted by H_c can be written as:

$$H_c(y, k, \lambda) = H(y, k, \eta) e^{(\delta-n)t} = U(c(y,k)) + \lambda(y - nk)$$

where $\lambda = \eta e^{(\delta-n)t}$.

By the maximal principle:

$$\frac{\partial U}{\partial c}\frac{\partial c}{\partial y_j} = -\lambda_j$$

and from the assumption of perfect competition:

$$\frac{\partial U}{\partial c}\frac{\partial c}{\partial k_j} = w_j$$

where w_j is the product of price and rental price of good j. It follows that:

$$\dot{k}_j = y_j - nk_j$$

$$\dot{\lambda}_j = -\frac{\partial U}{\partial c}\frac{\partial c}{\partial k_j} + \delta\lambda_j$$

$$\lambda_j = -\frac{\partial U}{\partial c}\frac{\partial c}{\partial y_j}$$

Consider the case $U' = 1$. It follows that:

$$\dot{k}_j = y_j(k,p) - nk_j$$

$$\dot{p_j} = -w(k,p) + \delta p_j.$$

To find the stability of the stationary point, we consider the jacobian matrix of this system:

$$J = \begin{pmatrix} \frac{\partial y}{\partial k} - nI & \frac{\partial y}{\partial p} \\ -\frac{\partial w}{\partial k} & -\frac{\partial w}{\partial p} + \delta I \end{pmatrix}$$

Exercise 10 *Consider a Cobb-Douglas technology and assume perfect competition* ($\frac{\partial w}{\partial k} = 0$),

1. *Show that depending of the value of δ there exists the possibility of a Hopf bifurcation, implying that closed orbits arise in a neighborhood of the fixed point $\dot{y} = \dot{p} = 0$.*

2. *Justify the following statement:* the usual argument in justifying governmental interventions into the markets processes point out in this case. *To do this consider the possibility or a political institution, influencing the discounting rate. f*

3. *Observe that, in spite of being considered fluctuations as non-optimal, it can appear naturally like result of an optimization process.*

13.10 Stationary equilibria and wealth distributions

The aim of this example is to analyze the effect of the initial wealth distribution on the dynamics of equilibria in a continuous time model. A similar problem is considered in [Ghigliano, Ch.; Sorger, G.].

We consider a continuous-time model of one sector economy in which at time $t \in [0, \infty)$ output is produced from capital $K(t)$, and labor $L(t)$ by the Cobb-Douglas technology

$$Y(t) = K(t)^{\alpha}L(t)^{1-\alpha}.$$

Output is the numerary good and we denote by $r(t)$ and $w(t)$ the rate of capital and the wage rate at time t. At every instant t, the representative firm maximizes the profits:

$$\Pi(t) = Y(t) - r(t)K(t) - w(t)L(t).$$

Prices are taken as given.

We assume that there exist only two types of households and that the two types differ from each other only in their initial wealth. The number of households of type i is $n_i,\ i = 1, 2$. All households have identical preferences described by the utility functional

$$J = \int_0^\infty e^{-\rho t} U(c_i(t))dt,$$

we assume that the instantaneous utility function has the form:

$$U(c) = \frac{c^{1-1/\theta} - 1}{1 - 1/\theta} + \beta ln(1 - l),$$

where $\theta > 0$ and $\beta > 0$. We shall denote by $\delta > 0$ the constant depreciation rate of capital, and by $r(t) - \delta$ the interest rate. Let $k_i(t)$ be the wealth of type i households, and let k_{i0} the exogenously given initial wealth of this household. With this notations the intertemporal budget constraint of a type i household can be written as:

$$\dot{k}_i(t) = [r(t) - \delta]k_i(t) + w(t) - c_i(t), \quad k_i(0) = k_{i0}$$

$$\lim_{t\to\infty} e^{-\int_0^t (r(s)-\delta)ds} k_i(t) = 0$$

whereby the first equation describe the wealth accumulation while the second one is a transversality condition. The factor markets are in equilibrium if:

$$K(t) = n_1 k_1(t) + n_2 k_2(t), \qquad L(t) = n_1 l_1(t) + n_2 l_2(t). \tag{41}$$

The output market is in equilibrium if:

$$Y(t) = \dot{K}(t) + \delta K(t) + C(t). \tag{42}$$

Profit maximization under perfect competition implies that capital and labor earn theirs marginal products, this yields:

$$r(t) = \alpha K(t)^{\alpha-1} L(t)^{1-\alpha}, \quad w(t) = (1 - \alpha)K(t)^\alpha L(t)^{-\alpha}$$

The first order optimality conditions for the optimization problem of a type i household are:

$$c_i(t)[w(t)/\beta]^\theta [1 - l_i(t)]$$

and

$$\frac{\dot{c}_i(t)}{c_i(t)} = \theta[r(t) - \rho - \delta].$$

The condition of Pareto optimality, for this problem implies, following the Negishi approach, see [Accinelli, E] . that $c_1/c_2 = \lambda_1/\lambda_2$ where λ_i is the welfare weight of corresponding to each agent of type i.

The per-capita evolution of the capital stock is given by the equation:

$$\dot{k} = k^\alpha - c(t) - (n + \delta)k$$

where n denotes the rate of growth of the labor. The stationary state is given by:

$$\dot{k} = 0, \quad and \ \dot{c} = 0.$$

Let $\bar{k}_i$ be the per-capita capital of type i households, so,

$$\dot{k} = \frac{n_1\dot{\bar{k}}_1 + n_2\dot{\bar{k}}_2}{n_1 + n_2}$$

then in the stationary state we should have the following equality:

$$\dot{\bar{k}}_1 = \frac{n_1}{n_2}\dot{\bar{k}}_2, \quad \forall t.$$

This implies that the initial distribution of wealth must satisfy this equation.

Now suppose that the government equalizes the wealth levels of all households by means of a redistribution. Due to the fact that after the redistribution all households are identical, they solve the same optimization problem. Since the utility is a strictly concave function, the solution is unique, all households choose the same consumption path, this implies $\lambda_1 = \lambda_2 = \frac{1}{2}$, and make the same same labor supply decisions. This implies that the economy is out of the stationary state.

On the other hand stationarity implies that

$$c = \frac{n_1c_1 + n_2c_2}{n_1 + n_2}$$

then, the welfare weights for the stationary equilibrium are $\lambda_1 = \frac{n_1}{n1+n_2}$ and $\lambda_2 = \frac{n_2}{n1+n_2}$

13.11 Appendix: The Solow growth model

Although this section has nothing to do with optimal control, we conisder that it plays an important roll in the understanding of the neo-classical aggregate growth model that we have considered in previous sections. The Solow model focuses on three variables, output Y, capital K, and labor L. The production function takes the form:

$$Y(t) = F(K(t), L(t))$$

were t denotes time.

The model's critical assumption is that the production function has constant returns to scale in its two arguments: $F(cK(t), cL(t)) = cF(K(t), L(t))$. So, we can write $y(t) = f(k(t))$ where $y = Y/L$ is the output per capita, and $k = K/l$ is the capital per capita.

We define s as the fraction of the output devoted to investment, and if in addition existing capital depreciates at rate λ the dynamic of the capital is given by:

$$\dot{K}(t) = sY(t) - \lambda K(t).$$

Observe that $C(t) = (1 - s)Y(t)$ and in this model s is given exogenously.

Considering capital per capita we obtain the equation:

$$\dot{k}(t) = sf(k(t) - (n + \lambda)k(t)$$

where the population grows at rate n. If we consider $\dot{k}$ as a function of k and if we consider k^* as the value of k such that $\dot{k}(k^*) = 0$, we obtain that, if initial capital $k(0) < k^*$ then $\dot{k}$ is positive and k converges to k^*, and, if initial capital $k(0) > k^*$, then $\dot{k}$ is negative, and remain constant $k(t) = k^*$ if $k(0) = k^*$.

The parameter of the Solow modelmost likely to be affected by plicy is the saving rate s. Since k converges to k^*, it is natural to ask how variations in this parameter affect the model.

1. **The impact on capital.** The increase of s shifts the actual line investment-capital, $\dot{k}(k)$ upward, and so k^* rises. But k does not immediately jump to the new value of k^*. Thus k begin to rise until reaches the new value of k^*. A permanent increase in the saving rate produces a temporary increase in the growth rate of capital per worker, k increase for a time but eventaully it increases to the the new k^*.

2. **The impact on consumption.** Consider now the impact on consumption. Since:

   $$c^* = f(k^*(t) - (\lambda - n)k^*(t),$$

   thus

   $$\frac{\partial c^*}{\partial s} = f'(k^*(s, n, \lambda) - (n + \lambda)\frac{\partial k^*(s, n, \lambda)}{\partial s}. \tag{43}$$

We know that the increase in s raises k^*. Thus whether the increase raises or lowers consumption in the long run depends on whether the marginal product of capital $f'(k^*(s,n,\lambda))$ is more or less than $n+\lambda$.

3. **Impact on output.** The long run effect of a rise in saving on output is given by:

$$\frac{\partial y^*}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,\lambda)}{\partial s}. \tag{44}$$

From the condition $\dot{k}(k^*) = 0$ we obtain:

$$sf(k^*(s,n,\lambda) = (n+\lambda)k^*(s,n,\lambda)$$

taking derivatives it follow that:

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+\lambda)+sf'(k^*)}$$

Multiplying both sides of (44) by s/y^* and from these equalities we obtain that

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

where $\alpha_k(k^*) = k^*f'(k^*)f(k^*)$.

If markets are competitive and there are no externalities, capital earns its marginal product. In this case, the total amount received by capital on the balanced growth path is $k^*f'(k^*)$ and if capital earns its marginal product, the share of total income that capital earn is α_k.

In most countries, the share of income paid to capital is above one-third. It follows that the elasticity of output with respect to the saving rate is one half. For example, a 10 percent of increase in saving rate (from 20 to 22 percent for instance) raises output per worker in the long run about 5 percent. So the impact of a substantial change in saving rate on output is modest. Then the main differences in output per worker between different countries on time in one country, have not its origin in the saving rate. The Solow model identifies two possible sources of variation;

- Capital per worker K/L and
- effectiveness of labor.

Following [Romer, D.], it is not possible to explain these differences in income between countries on the basis of differences in capital. The requires differences in capital are far too large, and there is not evidence of such differences in capital stocks, Then the only possibility is to consider differences in *effectiveness of labor.* This is represented by a new variable A in the model, this variable can be interpreted as the education and skills of labor forces. Now the production function takes the form $Y = F(K, A(t)L(t))$. Attributing differences in standards of living to differences in the effectiveness of labor does not require huge differences in capital or in rates of return.

14 The proof of the Pontriaguin M.P.

In this section we prove the maximal principle for the general case of nonlinear autonomous control system with moving targets and finite or infinite time horizon. We shall give the prove of theorem (54). The PMP is a necessary condition for an extremal controller and as we have shown in section (9), the maximal principle, together with the transversality conditions, is a necessary criterion satisfied by an optimal controller.

As we saw in some cases the attainable set is a convex set, but unfortunately this is not true in general. This is the main difficulty to obtain a necessary condition for an extremal controller. Essentially what needs to be proved is that there is a supporting hyperplane for this set. To overcome this difficulty, this set is replaced by a cone W with nonempty interior which is a subset of $K(t)$. Then we prove the PMP considering an hyperplane π in the vertex of this cone.

We start considering the basic geometric properties of the set of attainability. Next we prove that all extremal control satisfies the maximal principle (we shall show only an sketch of the proof), and finally we shall give the proof of the P.M.P.

Let us consider a non linear process defined by a differential system in R^n

$$\dot{x} = f(x, t, u), \quad (\mathcal{L}) \tag{45}$$

where f is in C^1 in R^{n+1+m}. The admissible controllers on the specified interval $t_0 \leq t \leq T$, is a family $\mathcal{F}$ of measurable $m-$vector functions. The initial point x_0 lies in a compact set $X_0 \in R^n$ and we assume that each response $x(t, x_0, t_0) = x(t)$ for $u(t) \in \mathcal{F}$ exists on the interval $t_0 \leq t \leq T$.

Suppose for each $u(t) \in \mathcal{F}$ there is a bound

$$|x(t, x_0, t_0)| < b, \ and \ |f(x, t, u(t))| + \left| \frac{\partial f}{\partial x}(x, t, u(t)) \right| < m(t),$$

with $\int_{t_0}^{T} m(t)dt < \infty$. Then there exists an unique response $x(t, x_0, t_0)$. In this case we say that the control $u(t)$ *admits a bound* for the response. If the bound B and the integrable function $m(t)$ can be chosen independently of the controller u, then the problem $\{(\mathcal{L}), x_0, \mathcal{F}, t_0, T\}$ has a uniform bound.

14.1 Geometric properties of the attainable set

1. If $\{(\mathcal{L}), x_0, \mathcal{F}, t_0, T\}$ has a uniform bound, then $\bar{K(t)}$ is a compact, continuously varying set in R^n for $t_0 \leq t \leq T$.

2. If there exists an uniform bound b such that $|x(t, x_0, t_0)| < b$, for all responses. and the set $V(x,t) = \{F(x,t,u(: u \in \Omega)\}$, the *set of velocity vectors* is convex, then:$k(\bar{t}))$ is a compact, continuously varying set in R^n for $t_0 \leq t \leq T$. Here Ω is a compact set.

   See [Lee, E.; Markus, L.] 243 .

We define regular point of a control u, as a point τ at which:

$$\int_{\tau-\epsilon}^{\tau} |f(x(t), u(t)) - f(x(\tau), u(\tau))| dt = o(\epsilon)$$

so,

$$\int_{\tau-\epsilon}^{\tau} f(x(t), u(t)) dt = f(x(\tau), u(\tau))\epsilon + o(\epsilon).$$

Almost all points are regular. If the reader prefers to avoid meassure theory, he can assume piecewise constant controls, and use non-jump points as regular points.

For convenience we first prove the maximal principle for an autonomous system in R^n, $\dot{x} = f(x,u)$, the set of controllers Ω is not necessarily compact.

14.2 Displacement of tangent spaces along $\bar{x}(t)$

Let $\bar{u}(t)$ be an admissible control with response $\bar{x}(t)$ on $0 \leq t \leq T$. Along the flow $\dot{x}(t) = f(x, \bar{u}(t))$ there is a transport or displacement vector v along $\bar{x}(t)$:

$$\dot{v} = \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t))v.$$

Let $\eta(t_1)$ be the direction of the normal to the hyperplane π_{t_1} at $x(t_1)$ defines the solution $\eta(t)$ of the adjoint system: $\dot{\eta} = -\eta \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t))$ where $\eta(t_1)v_{t_1} = 0$.

Then $\eta(t)v(t) = 0$ for all $v(t) \in \pi_t$, since

$$\frac{d}{dt}[\eta(t)v(t)] = \dot{\eta}v + \eta\dot{v} = 0$$

Thus each nontrivial solution $\eta(t)$ of the adjoint system define a parallel displacement of the hyperplane π_t along $\bar{x}(t)$, and every such parallel field π_t arises in this way.

A tangent vector x_1 in R^n is determined by a differentiable curve $x = \phi(\epsilon)$ with $\phi(0) = x_1$. Let $v_1 = \dot{\phi}(0)$ be the components of this tangent vector. (Actually, a tangent vector can be defined as a class of differentiable curves all of which have the same tangent components, v_1 at x_1.)

Let $x(t,z)$ be the solution of $\dot{x}(t) = f(x, \bar{u}(t))$ such that $x(t_1, z) = z$.

If $\phi(\epsilon) = x(t_1, \phi(\epsilon))$ describes a tangent vector at $x_1 = \bar{x}(t_1)$, we define the displaced $A_{t_2 t_1}\phi(\epsilon) = x(t_2, \phi(\epsilon)$. And we define the displacement of v_1, the tangent vector of this curve at $x(t_1)$ as:

$$v_2 = A_{t_2 t_1} v_1 = \frac{d}{d\epsilon}[A_{t_2 t_1}\phi(\epsilon)]|_{\epsilon=0} = \frac{\partial x}{\partial z}(t_2, z)|_{\epsilon=0}\dot{\phi}(0).$$

Thus the real n-dimensional tangent space based at $x_1 = \bar{x}(t_1)$ is displaced onto the tangent space based at $x_2 = x(t_2)$ by the linear transformation $A_{t_2 t_1}$, which is described by the matrix $\frac{\partial x}{\partial z}|(t_2, x_1)$. But:

$$\frac{d}{dt}\frac{\partial x}{\partial z}|_{t,x_1} = \frac{\partial}{\partial z} f(x(t_1, z)\bar{u}(t)|_{z=x_1} = \frac{\partial}{\partial x} f(x(t_1, z)\bar{u}(t))|_{x=x(t_1,x_1)} \frac{\partial}{\partial z} x(t_1, z)|_{z=x_1}$$

and so $(\partial x/\partial z)(t, x_1)$ is the fundamental matrix solution of the variational differential system $\dot{v} = \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t))v$ with $(\partial x/\partial z)(t_1, x_1) = I$.

Therefore the displaced vector $v(t) = A_{tt_1}\dot{\phi}(0)$ is the solution of the variational equation $\dot{v} = \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t))v$ with $v(t_1) = \dot{\phi}(0)$.

14.3 The perturbation cone

The key idea is to perturb a basic control $\bar{u}(.)$ by changing its value to an admissible vector u over any small time interval.

We pick a time $0 < t_1 < T$, and near this time we perturb the controller $u(t)$ changing its value to some constat $u_1 \in \Omega$. For $t_1 - \epsilon l_1 < t < t_1$ change the control from the optimal u^* to another admissible control u_1. Here l_1 is a non negative constants. After $t = t_1$ we return back to the optimal control u.

$$u_\epsilon = \begin{cases} u^*(t), & t \notin [t_1 - \epsilon l_1, t_1] \\ u_1 & t \notin [t_1 - \epsilon l_1, t_1] \end{cases}$$

Here the perturbation data are $\pi_1 = \{t_1, l_1, u_1\}$ for $0 < t_1 < T, l_1 \geq 0$, and $v_1 \in \Omega$.

Select now a distinct set of times $t_1 < ... < t_s$ an a set of elementary transformation $\pi_i = \{t_i, \lambda l_i, u_i\}, i = 1, ...s$. Where the perturbed function $u_{\pi_i}(t, \epsilon)$ is a well defined admissible controller with response $x_{\pi_i}(t, \epsilon)$ initiating at $x_{\pi_i}(0, \epsilon) = x_0$. Moreover, it is easy to see that :

$$\lim_{\epsilon \longrightarrow 0} x_{\pi_i}(t, \epsilon) = x^*(t),$$

uniformly on $0 \leq t \leq T$.

The next step is to transfer a convex combination of these elementary perturbations along the trajectory $\bar{x}(t)$ from the starting point at time t_1 to another time t using the control $\bar{u}$ for both of them.

In order to appreciate the nature of this process, We shall do the following considerations. Let $u_{\pi_i}(t,\epsilon)$ be an elementary perturbation of $u^*(t)$ at $\pi_i = \{t_i, l_i, u_i\}$. Then the corresponding response $x_{\pi_i}(t,\epsilon)$ defines a tangent vector at t_i by the curve $\phi(\epsilon) = x_{\pi_i}(t_i,\epsilon)$. Namely,

$$\dot{\phi}(0) = \lim_{\epsilon \to 0}[x_{\pi_i}(t,\epsilon) - x(t_i)] = f(x^*(t_i), u_i) - f(x^*(t_i), u^*(t_i)]l_i.$$

This follows from the estimate:

$$x_{\pi_i}(t_i\epsilon) = x^*(t_i - l_\epsilon) + \int_{t_i - l_i\epsilon}^{t_i} f(x_{\pi_i}(t,\epsilon), u_i)dt,$$

or

$$x_{\pi_i}(t_i,\epsilon) = x^*(t_i) - f(x^*(t_i), u_i)l_1\epsilon + f(x^*(t_i), u^*(t_i)]l_i\epsilon + o(\epsilon).^7$$

The tangent vector at $\bar{x}(t_i)$,

$$v_{\pi_i}(t_i) = [f(x^*(t_i), u_i) - f(x_i^*(t_i), \bar{u}(t_i))]l_i$$

is called the elementary perturbation vector for the data $\pi_i = left\{t_i, l_i, u_i)$. Note that the data $\{t_i, \lambda l_i, u_i\}$ for $\lambda \geq 0$ yield a perturbation vector $\lambda v_{\pi_1}(t_1)$, and hence the elementary perturbation vectors fill a cone in the tangent space at $x(t_i)$, that is, with each point of the cone the entire ray trough that point also lies in the cone. As we saw before, the law by which this perturbation is transferred along the trajectory $\bar{x}(t)$ is given bay $\dot{v}_\pi = A_{tt_1}v_\pi$ where: $A = f_x(\bar{x},\bar{u})$.

Until now, we have considered a single perturbation from the optimal solution. We can find other perturbations by changing the value of the perturbed control u_1 and time t at which we make the perturbation.

If we select a distinct set of times $t_i : t_0 < t_1 < ... < t_p < T$ and perturb $u^*(.)$ near each t_i by u_i as described above, then for any $t > t_p$ the resulting response can be written:

$$(*) \quad x_\pi(t,\epsilon) = +\epsilon(\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n) + o(\epsilon)$$

The fundamental perturbation formula (*) shows that any convex combination of elementary perturbations vectors (at distinct times) define a point $\bar{x}(\bar{t}) + \epsilon v_\pi$, which lies in the set of attainability $K(\bar{t})$ within an error of $o(\epsilon)$. Where $v_\pi = \sum \lambda_i v_{\pi_i}(t)$ is a convex combination of elementary perturbation. Here $\pi_i = \{t_i, l_i, u_i\}$ with $0 \leq t_i \leq \bar{t},\ l_i > 0$.

[7] This follows from:

$$x_\pi(t,\epsilon) = x(t_i - l_i\epsilon) + \int_{t_i - l_i\epsilon}^{t} f(x_\pi(t,\epsilon), u_\pi)dt, \quad and \quad x(t,\epsilon) = x(t_i - l_i\epsilon) + \int_{t_i - l_i\epsilon}^{t} f(\bar{x}(t,\epsilon), \bar{u})dt,$$

then

$$x_\pi(t,\epsilon) - x(t,\epsilon) = \int_{t_i - l_i\epsilon}^{t} [f(x_\pi, u_\pi) - f(\bar{x},\bar{u})dt] = [f(x_\pi, u_\pi) - f(\bar{x},\bar{u})dt].$$

Let K_t be the tangent perturbation cone at any $0 \leq t \leq T$, it is the smallest closed convex cone in the tangent space at $\bar{x}(t)$ containing all parallel displacements of elementary perturbation vectors form Lebesgue (regular) times t_1 on $0 \leq t_1 \leq t$.

Note that $A_{\bar{t}t} K_t \subset K_{\bar{t}}$ for $t < \bar{t}$, and $K_{\bar{t}} = \cup_{0<t<\bar{t}} A_{\bar{t}t} K_t$ in particular for the final limit cone where

$$\bar{t} = T = K_T = \cup_{0<t<T} A_{Tt} K_t$$

Definition 75 *Let $v_1, ..., v_n$ be independent vectors in K_t, each arising as a convex combination of elementary perturbation vectors $v_{\pi_i}(\bar{t}) \quad i = 1, 2...\pi_s$. An* **elementary simplex cone, C** *consists of all convex combination of vector $v_1, ..., v_n$.*

The fundamental perturbation formula (*) asserts the existence of a response

$$x(t, \epsilon, \lambda) = x^*(t) + \epsilon(\lambda_1 v_1 + ... + \lambda_n v_n) \in \mathcal{C}.$$

Lemma 76 *Let v be a vector interior to K_t (the attainable set at t). Then there exists an elementary simplex cone $\mathcal{C}$ which contains v in its interior.*

This result of approximation can be founded in [Lee, E.; Markus, L.] 251.

Lemma 77 *Let v be a nonzero vector interior to K_t. Then there exists an elementary simplex cone $\mathcal{C}$ in K_t such that:*

1. *$\mathcal{C}$ contains v in its interior.*
2. *$\mathcal{C}$ lies interior to $K(t)$ as a macroscopic cone (that is, a truncation of $\mathcal{C}$ minus its vertex lies interior to $K(t)$ near $\bar{x}(t)$.)*

14.4 The proof of the maximal principle.

In this section we prove that all extremal controller verify the maximal principle. That is a response end points belongs to the boundary of the attainable set only if maximal condition holds.

To prove the theorem, we proceed as follows. First we establish the effects of perturbations of a control on the end points of the corresponding trajectory. Certain convex sets of perturbations are then considered and it is shown that, to first order, the end points of the perturbed trajectories generate a cone. As a consequence of the optimality assumptions, this cone is separated from certain hyperplane. The analytic consequences of this separation constitue theoreme (54).

14.5 Existence of extremal controller without magnitude restraints

We begin with an arbitrary control restraints $u(t) \in \Omega$ which need not be compact, and there are neither state constraints nor on target constraints . If $\bar{u}(t)$ is extremal, then there exists a nontrivial adjoint response such that $\bar{u}(t)$ satisfies the maximal principle.

Theorem 78 *Consider the autonomous control process in* R^n

$$\dot{x} = f(x,u) \qquad (\mathcal{S})$$

with $f(x,u)$ *and* $\partial f/\partial x(x,u)$ *continuous in* R^{n+m}. *Let* $\mathcal{F}$ *be the family of all measurable controllers* $u(t)$ *on* $0 \leq t \leq T$ *that satisfy the restraint* $u(t) \in \Omega \subset R^m$ *and admit a bound for the response* $x(t,x_0)$ *initiating at t he point* x_0. *Let* $\bar{u}(t) \in \mathcal{F}$ *have a response* $\bar{x}(t)$ *with* $\bar{x}(T)$ *on the boundary of the set of attainability* $K(T)$. *Then there exists a non trivial adjoint response* $\bar{\eta}(t)$ *of*

$$\dot{\eta} = -\eta \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t)) \qquad (\mathcal{A})$$

such that the maximal principle obtains, that is:

$$\bar{H}(\bar{\eta}, \bar{x}, \bar{u}) = M(\bar{\eta}, \bar{x}).$$

Further, if $\bar{u}(t)$ *is bounded* $M(\bar{\eta}^*, \bar{x})$ *is constant everywhere.*

Here the Hamiltonian function is:

$$H(\eta, x, u) = \eta f(x,u) = \eta_1 f_1(x,u) + ... + \eta_n f_n(x,u)$$

and

$$M(\bar{\eta}, \bar{x}) = \max_{u \in \Omega} \bar{H}(\bar{\eta}, \bar{x}, \bar{u})$$

Proof:

- Since $\bar{x}(T)$ lies on the boundary of $K(T)$, there is a sequence of points $\{P_n\}$ outside $K(T)$ such that $P_n \to \bar{x}(T)$ and the unit vectors along the segments $\bar{x}(T)$ to P_n approach a limit unit vector $w(T)$ at $\bar{x}(T)$.
- Now $w(T)$ cannot be an interior vector in the perturbation cone K_T; this cone minus its vertex lies in the interior of $K(T)$ (see 76), and this contradicts the assumption that all P_n lie outside $K(T)$.

- Thus there exists a hyperplane $\pi(T)$ at $\bar{x}(T)$ such that $\pi(T)$ separates $w(T)$ from K_T. Let $\bar{\eta}(T)$ be the exterior unit normal to $\pi(T)$ at $\bar{x}(T)$ and define $\bar{\eta}(t)$ as a solution of nonlinear system $\mathcal{A}$.

Then $\bar{\eta}(T)\xi(T) \leq 0$. Since $\xi(t)$ verifies $\dot{\xi} = \frac{\partial}{\partial x} f(x,u)\xi$ (is a parallel displacement) it follows that

$$\bar{\eta}(T)\xi(T) = \bar{\eta}(t)\xi(t) \leq 0, \quad for\ all\ t \leq T,$$

Suppose that the maximal principle fails, ,that is

$$\bar{H}(\bar{\eta}(t), \bar{x}(t), \bar{u}(t)) < \bar{H}(\bar{\eta}(t), \bar{x}(t), u_1(t))$$

for some $u_1 \in \Omega$ on some duration of positive measure on $0 \leq t \leq T$. Let t_1 on $0 < t_1 T$ be a Lebesgue time for $f(\bar{x}(t), \bar{\eta}(t))$ when

$$\bar{\eta}(t_1) f(\bar{x}(t_1), \bar{u}(t_1)) < \bar{\eta}(t_1) f(\bar{x}(t_1), u_1)).$$

Consider the elementary perturbation vector

$$v(t_1) = [f(\bar{x}(t_1), u_1) - f(\bar{x}(t_1), \bar{u}_{t_1})].$$

Then the denial of the maximal principle yields: $\bar{\eta}(t_1)v(t_1) > 0$, which contradicts the assertion that $\bar{\eta}(t)v(t) > 0$, for all t and all $v(t) \in K_t$.

The final result that we have to establish is that the Hamiltonian is constant along the optimal trajectory.

- When there is no restriction on the magnitude of the controllers and the optimal control is regular the maximal principle shows that along the optimal trajectory: $\frac{\partial H}{\partial u} = 0$.

It follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \eta}\dot{\eta} = 0$$

where we have used the state and coestate equations:

$$\dot{x} = \frac{\partial H}{\partial \eta}, \quad \text{and} \quad \dot{\eta} = -\frac{\partial H}{\partial x}.$$

However, we know that the optimal control is often discontinuous, therefore we need a more refined argument.

- The general case. We show that $M(\bar{\eta}(t), \bar{x}(t))$ has a zero derivative everywhere on $0 \leq t \leq T$.

Here we assume that $\bar{u}(t)$ is bounded, that is $|\bar{u}(t)| \leq \beta$ on $0 \leq t \leq T$. Let:

$$m(\eta, x) = \max_{|u| \leq \beta, u \in \Omega} H(\eta, x, u),$$

so $M(\eta, x) \geq m(\eta, x)$ but $M(\bar{\eta}(t), \bar{x}(t)) = m(\bar{\eta}(t), \bar{x}(t))$ almost everywhere on $0 \leq t \leq T$.

We first show that $m(\bar{\eta}(t), \bar{x}(t))$ is constant on $0 \leq t \leq T$. For (η, x) is a compact set Q in $R^n \times R^n \times R^M$ containing $(\bar{\eta}(t), \bar{x}(t))$ and all $|u| \leq \beta$. We obtain for any two points (η, x, u) and (η', x', u),

$$|H(\eta, x, u) - H(\eta', x', u)| \leq kd$$

where $d = |\eta - \eta'| + |x - x'|$ and k is a Lipschitz constant majorizing $|f(x, u)|$ and $|\eta \frac{\partial f}{\partial x}(x, u)|$ in Q

Let u and u' in Ω with $|u| \leq \beta, |u'| \leq \beta$ be selected so that:

$$m(\eta, x) = H(\eta, x, u) \text{ and } m(\eta', x') = H(\eta', x', u'),$$

Then

$$H(\eta, x, u') \leq H(\eta, x, u) \text{ and } H(\eta', x', u) \leq H(\eta', x', u').$$

Therefore, we compute on Q

$$-kd \leq H(\eta, x, u') - H(\eta', x', u') \leq H(\eta, x, u) - H(\eta', x', u') \leq H(\eta, x, u) H(\eta', x', u') \leq kd$$

and

$$|m(\eta, x) - m(\eta, x')| \leq kd.$$

Hence $m(\eta, x)$ is Lipschitz continuous in Q and so $m(t) = m(\bar(\eta)(t), \bar{x}(t))$ absolutely continuous on $0 \leq t \leq T$. (See Appendix below).

Let $0 \leq \tau \leq T$ be a point at which $m(t)$ and $\bar{x}(t)$ and $\bar{\eta}(t)$ all have derivatives. For $t' > \tau$ we compute: $m(t') \geq H(\bar{\eta}(t'), \bar{x}(t'), \bar{e}(\tau))$ and: $m(t') - m(\tau) \geq H(\bar{\eta}(t'), \bar{x}(t'), \bar{e}(\tau)) - H(\bar{\eta}(t'), \bar{x}(\tau), \bar{u}(\tau))$

$$+H(\bar{\eta}(t'), \bar{x}(\tau), \bar{u}(\tau)) - H(\bar{\eta}(\tau), \bar{x}(\tau), \bar{u}(\tau)),$$

then

$$\lim_{t' \to \tau} \frac{m(t') - m(\tau)}{t' - \tau} = \frac{dm}{dt}|_{t=\tau} \geq \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \eta}\dot{\eta}|_{t=\tau} = 0$$

Using $t' \leq \tau$ we compute $\frac{dm}{dt}|_{t=\tau} \leq 0$ so $\frac{dm}{dt}|_{t=\tau} = 0$ almost everywhere. Since $m(\bar(\eta)(t), \bar{x}(t))$ is absolutely continuous on $0 \leq t \leq T$ with a zero derivative, then it is a constant m everywhere on $0 \leq t \leq T$.

The reader should note that the PMP is valid for any extremal solution, that is any solution that lies on the boundary of attainable set at time t_1, in the extended state space. For linear processes a controller satisfies the maximal principle if and only if $u(t)$ is extremal. For nonlinear process the maximal principle does not guarantee that $u(t)$ steers $x(t)$ to the boundary of the set of attainability. Here the concept of frontier of $K(t)$ as developed for linear process is not significant. The frontier can not be convex, nor even simply connected. For an example where a point (x, u) satisfies the PMP but is not extremal see [Lee, E.; Markus, L.] 257.

14.6 Appendix: Definitions and theorems

The following definitions and theorems were considered in the proofs of the PMP.

Definition 79 *Let $a, b \in R, a < b$. A function $f : [a, b] \to R$ is called absolutely continuous in $[a, b]$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all finite partition $\{(x_i, y_i)\}\, i = 1^n$ of $[a, b]$ such that*

$$\sum_{i=1}^{n}(y_i - x_i) < \delta,$$

we have that:

$$\sum_{i=1}^{n}|f(y_i) - f(x_i)| < \epsilon.$$

Definition 80 *Let $f(t)$ a real function $t \in [a, b]$, define the total variation*

$$var f(t) = sup \sum_{j=0}^{k} f(t'_{j+1}) - f(t'_j)|$$

where $t'_0 < t'_1, ..., t'_k$ is an arbitrary finite set of points, the supremun is computed over all such finite sequences in $[a, b]$. We say that the function $f(t)$ is of bounded variation in $[a, b]$ if $var f(t) < \infty$.

Theorem 81 *If $f : [a, b] \to R$ if an absolutely continuous function, then f is of bounded variation.*

Corollary 82 *All absolutely continuous function $f(t)$ on $[a, b]$ have a derivative almost everywhere.*

Theorem 83 *Every Lipschitz continuous function an $[a, b]$ is absolutely continuous on this interval.*

15 Variational Calculus and its relations with Control Theory.

In this section we investigate the relationship between the maximal principle and the first order conditions in the calculus of variations. The calculus of variations, like ordinary calculus, requires for its applicability the differentiability of the functions that enter in the problem. More important only interior solutions can be handled. We begin showing some typical problems of the variational calculus and giving the main rules to solve them. In spite of being less general than the optimal control one, this approach has value from itself . This approach allows us to solve in an efficient way difficult problems of optimization, provided that we do not need to obtain the control. Also in this section we examine the optimal control problem from the point of classical calculus of variations and conversely, in the last subsection of this section, we show how the classical first order necessary conditions in the calculus of variations can be obtained from the maximal principle.

On the other hand, if the set of controllers is not restricted, and the target is not a specified point $x(t_f)$, we can obtain from this approach necessary conditions for an extremal. In other case the geometrical development is more general an completely correct in all mathematical details

For instance, the following simplfied version of the isoperimetric problem is a characteristic problem that can be solved using variational methods. We shall show this problem farther on.

15.1 Example: A typical variational problem

For instance consider the problem of inding an extremum (maximum or minimum) in the space of $C^1([t_0, t_1])$ function to the following functional:

$$\mathcal{V} \qquad \mathcal{B}(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt + l(x(t_0)) + l(x(t_1))$$

in a finite interval $[t_0, t_1]$, where $L : \Omega \to R$, and such that the derivatives

$$L_{\dot{x}}(t) = \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))$$

and

$$L_x(t) = \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))$$

are continuous in $[t_0, t_1]$, and $\Omega \subseteq R^3$.

A rule for the resolution: If a function $\bar{x}$ is an extremum then it solves:

1. The *Euler equations*

$$-\frac{d}{dt}\bar{L}_{\dot{x}}(t) + \bar{L}_x(t) = 0$$

2. The *Transversality Conditions*

$$\bar{L}_{\dot{x}}(t_0) = \bar{l}_{x_0}, \quad \bar{L}_{\dot{x}}(t_1) = \bar{l}_{x_1}$$

3. To find the admissible extremals solve this equations:

4. Show that one of these extremals is the solution of the problem.

Example 84 *The length of an arc through* (x_0, y_0) *and* (x_1, y_1) *is given by:* $l[y(x)] = \int_{x_0}^{x_1} \sqrt{1+y'^2}dx$*. The extremal arc is a stright line* $y = C_1x + C_2$ *where the constants* C_1 *and* $C - 2$ *are determined from the initial and final conditions.*

Example 85 *The time that a point describing an arc* $y(x)$*, joining the points* $A(x_0, y_0)$ *and* $B(x_1, y_1)$ *whith velocity* $v(y') = \frac{ds}{dt}$ *depends only on* y'*, is given by the equation*

$$t[y(x)] = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v(y')}dx.$$

The extremal curve is an straight line.

Example 86 *Show that there exists a minimun but not a maximun to the following extremal problem:*

$$\mathcal{B}(x) = \int_0^1 ((\dot{x}(t))^2 - x(t))dt + x^2(1)$$

The necessary conditions are:

1. $-\frac{d}{dt}\bar{L}_{\dot{x}} + \bar{L}_x = 0, \leftrightarrow \ 2\frac{d}{dt}\dot{x} + 1 = 0.$

2. $$\bar{L}_{\dot{x}}(t_0) = \bar{l}_{x_0}, \leftrightarrow \ \dot{x}(0) = 0$$
$$\bar{L}_{\dot{x}}(t_1) = \bar{l}_{x_1}, \leftrightarrow \ \dot{x}(1) = -x(1).$$

3. There is only one extremal: $\bar{x}(t) = -\frac{t^2}{4} + \frac{3}{4}$.

4. We shall show now, that this extremal is a local minimum.

Consider $h \in C^1([t_0, t_1])$ and: $\mathcal{B}(\bar{x} + h) - \mathcal{B}(\bar{x}) =$

$$\int_0^1 2\dot{\bar{x}}\dot{h}dt + \int_0^1 \dot{h}^2dt - \int_0^1 hdt + 2\bar{x}(1)h(1) + h^2(1).$$

Then we integrate by parts and since $\bar{x}(t) = -\frac{t^2}{4} + \frac{3}{4}$, it follows that:

$$\mathcal{B}(\bar{x} + h) - \mathcal{B}(\bar{x}) = \int_0^1 \dot{h}^2dth^2(1) \geq O.$$

Then the admissible extremal is a minimum, $S_{min} = \mathcal{B}(\bar{x}) = -\frac{8}{12}$ and it is easy to see that $S_{max} = \infty$.

The elemental **isoperimetric problem** consists is finding the curve enclosing the greatest area among all closed curves of a given length. (Isoperimetric means *with the same perimeter*). This symbolic expression is:

$$maxS(x,y) = \int_{t_0}^{t_1} xydt,$$

with the initial and terminal conditions :

$$x_s(t_s) = x_s,\ y(x_s) = y_s \ \ s = 1,2;$$

and the isoperimetric equation:

$$\int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} = l.$$

The most elemental isoperimetric problem is to solve the maximun for $S = \int_{t_0}^{t_1} ydx,\ \ y(x_0) = y_0 \ \ y(x_1) = y_1$, with the isoperimetric equation $\int_{t_0}^{t_1} \sqrt{1+\dot{y}^2}dx = l$, and given initial and final conditions.

We can solve this problem using the auxiliar functional:

$$L(x_0, x_1, y, \lambda) = \int_{t_0}^{t_1} y + \lambda\sqrt{1+\dot{y}^2}dx.$$

and using the Euler equation $F_y - \frac{\partial}{\partial x}F'_y = 0$.

15.2 A classical variational approach for the maximal principle

In this section we examine the optimal control problem from the point of view of the classical calculus of variations and without any discussion of continuity or differentiability. Consider the control process in R^n (x is a real n-vector):

$$\dot{x} = f(x,u), \ \ x(0) = x_0.$$

with controller $u(t) \in R^m,\ on\ \ 0 \le t \le 1$, and with fixed initial state $x(0) = x_0$, and with cost:

$$C(u) = \int_0^1 h(x,u)dt.$$

We impose no restraint on $u(t)$ **or** $x(t)$. If the controller $u(t)$ is restricted in magnitude, or the target $X(1)$ is specified, the variational techniques become much more complicated both from a formal viewpoint and from a technical viewpoint. For these reasons we shall not follow the classical calculus of variations.

Let $u^*(t)$ be an optimal controller minimizing $C(u)$ and let $x^*(t)$ be the corresponding optimal response.

Let $u(t,\epsilon) = u^*(t) + \epsilon\delta u(t)$ be a one parameter family of perturbations, with corresponding responses: $x(t,\epsilon) = x^*(t) + \epsilon\delta x(t) + o(\epsilon), \quad \delta x(0) = 0$.

Note that

$$u(t,0) = u^*(t) \quad and \quad \tfrac{\partial u}{\partial \epsilon}(t,0) = \delta u(t)$$

$$x(t,0) = x^*(t) \quad and \quad \tfrac{\partial x}{\partial \epsilon}(t,0) = \delta x(t)$$

The variation in cost is:

$$\frac{\partial C}{\partial \epsilon} = \delta C = \int_0^1 \left[\frac{\partial h(t)}{\partial x}\delta x(t) + \frac{\partial h(t)}{\partial u}\delta u(t) \right] dt.$$

Since $C(u(\cdot,\epsilon))$ is minimized at $\epsilon = 0$ we obtain: $\delta C(u^*) \equiv 0$, for all variations $\delta u(t)$. This necessary condition for the optimal controller will now be clarified.

The variation $\delta u(t)$ yields the response variation $\delta x(t)$, which satisfies the variational differential equation:

$$\delta \dot{x} = \frac{\partial f(t)}{\partial x}\delta x(t) + \frac{\partial f(t)}{\partial u}\delta u(t), \quad \delta x(0) = 0. \tag{46}$$

From (46) it follow that

$$\delta x(t) = \int_0^t \Phi(t)\Phi^{-1}(s)\frac{\partial f(s)}{\partial u}\delta u(s)ds.$$

where the fundamental matrix satisfies:

$$\dot{\Phi} = \frac{\partial f(t)}{\partial x}\Phi, \quad \Phi(0) = I.$$

Then

$$\delta C = \int_0^1 \left[\frac{\partial h(t)}{\partial x} \int_0^t \Phi(t)\Phi^{-1}(s)\frac{\partial f(s)}{\partial u}\delta u(s)ds + \frac{\partial h(t)}{\partial u}\delta u(t) \right] dt. \tag{47}$$

We introduce the notation:

$$\eta^*(t) = -\eta_0\Phi^{-1}(t) + \int_0^t \frac{\partial h(s)}{\partial x}\Phi(s)\Phi^{-1}(t)ds,$$

with the constant vector η_o so chosen that:

$$\eta^*(1) = -\eta_0\Phi^{-1}(1) + \int_0^1 \frac{\partial h(s)}{\partial x}\Phi(s)\Phi^{-1}(1)ds = 0,$$

This means that $\eta^*(t)$ is the unique solution of the adjoint variational differential equation

$$\dot{\eta} = -\eta\frac{\partial f}{\partial x} + \frac{\partial h}{\partial x}, \qquad \eta(1) = 0.$$

Now defining the Hamiltonian function of $2n+m$ real variables:

$$H(\eta, x, u) = \eta f(x,u) - h(x,u),$$

it follows that

$$\dot{\eta} = -\tfrac{\partial H}{\partial x}, \quad \eta(1) = 0$$

$$\dot{x} = \tfrac{\partial H}{\partial \eta}, \quad x(0) = x_0,$$

are satisfied by $\eta^*(t)$ and $x^*(t)$, when $u = u^*(t)$.

Using this notation we obtain:

$$\delta C = \int_0^1 - \left[\eta^* \frac{\partial f(t)}{\partial u} - \frac{\partial h(t)}{\partial u}\right] \delta u(t) dt.$$

Since $\delta C \equiv 0$ for all variation $\delta u(t)$ about the optimal $u^*(t)$, we find that

$$-\eta^*(t)\frac{\partial f(t)}{\partial u} - \frac{\partial h(t)}{\partial u} = 0$$

or

$$\frac{\partial H}{\partial u}(\eta^*(t), x^*(t), u^*(t)) \equiv 0.$$

A more detailed study of the variations about the minimizing controller $u^*(t)$ shows that $u = u^*(t)$ is a maximun rather than an arbitrary critical point of $H(\eta, x, u)$. That is

$$H(\eta^*(t), x^*(t), u^*(t)) = \max_{u \in R^m} H(\eta^*(t), x^*(t), u(t))$$

this is the maximal principle. The system of equations

$$\dot{x} = \frac{\partial H}{\partial \eta}, \ \ \dot{\eta} = -\frac{\partial H}{\partial x}, \ \ \frac{\partial H}{\partial u} = 0$$

are the Euler-Lagrange equations.

Remark 87 *If the controller is restricted in magnitude, or the target $x(1)$ is specified, the variational techniques become more complicated, for this reason we do not follow this viewpoint in our presentation.*

15.3 The Ramsey Model of Economic Growth

We consider here an economy in which a single homogeneous good is produced with the aid of capital $K(t)$ which may depend on time t, and labor L; the total output $Y(t)$ is either consumed or invested. Thus $C(t)$ is the total consumption. We have $Y(t) = C(t) + K'(t)$. It is assumed that there is not deterioration or depreciation of capital, and the production $Y(t)$ is a known function $Y = \Phi(K)$ of capital. We shall require that $C \geq 0, \Phi(K) \geq 0$, while K' may be positive or negative. Since there is no depreciation or deterioration, the capital can be consumed. In order to produce its consumption goods, society incurs disutility of labor $D(L)$, with nondecreasing marginal utility $D''(L) \geq 0$. The net social utility at time t is therefore; $U(C) - D(L)$.

We assume that the objective of any planing concerns the standard of living, that is we should try to maximize the global utility function

$$W = \int_{t_1}^{t_2} [U(C(t)) - D(L)]dt \tag{48}$$

here $C(t) = Y(t) - K'(t)$ and $Y(t) = \Phi(K(t), L)$. The time interval is finite, but we need not exclude $t_2 = +\infty$, infinite horizon. We assume that $U(C)$ is a smooth, positive nondecreasing function of C for $C \geq 0$.

However observe that the improper integral, for infinite time interval, does not contain a discount factor. This omission is not the result of neglect, it stems from the Ramsey's view that it is ethically undesirable for the current generation to discount utility of future generation. While this may be plausible on moral grounds, the absence of a discount factor implies difficulties for the convergence of the integral.

To overcome this difficulty in this case, Ramsey replaces (48) with the following substitute problem:

$$\begin{aligned} &\text{minimize} \quad \textstyle\int_{t_1}^{\infty} [B - U(C(t)) + D(L)]dt \\ &\text{subject to} \quad K(t_1) = K_1, \end{aligned} \tag{49}$$

where B (for Bliss) is a postulate maximun attainable level of net utility. Intuitively, an optimal plan should either take society to Bliss, or lead it to approach Bliss asymptotically. This substitution of (48) for (49) is referred as the Ramsey device, and is widely accepted as sufficient for convergence.

The solution of the model in a finite time interval $[t_1, t_2]$:

Replacing $C(t) = Y(t) - K'(t)$ in (48) we have the problem of variations concerning the maximun of

$$W = \int_{t_1}^{t_2} U(\Phi(K(t), L) - K'(t)) - D(L)dt$$

We consider the two obvious constraints, $K(t) \geq 0$ and $K' \leq \Phi(K(t))$. However, neither of both cases $K = 0$ and $K' = \Phi(K, L)$ should be taken into consideration. We shall see that an optimal solution will be in the interior of the domain.

Here we shall consider that the labor input is constant reducing the production function to $\Phi(K)$. We have a free problem of calculus of variation with:

$$f_0(K, K') = U(\Phi(K) - K') \quad f_{0K'} = -U'(\Phi(K) - K'),$$

and the Euler equation for this case, $f_0 + K'f_{0K'} = c$ becomes

$$U(\Phi(K, L) - K') - K'U'(\Phi(K, L) - K') = c, \tag{50}$$

where c is a constant.

If we consider the particular choice for the function $U(C) = U^* - \alpha(C - C*)^2$. for $=\leq C \leq C^*, \alpha > 0$ and $U(C) = U^*$ for $C \geq C^*$ (it can be said that the utility function U *saturates* at $C = C^*$.) Also we assume that the economy is bellow the point of saturation: $0 \leq C(t) < C^*$ for $t_1 \leq t \leq t_2$. We shall choose for the production a linear function: $Y = \Phi(K) = \beta K$, where β is a positive constant. The value of U^* is not relevant for the case on finite time interval, and we choose $U^* = 0$

Equation (50) becomes

$$-\alpha(\beta K - K' - C^*)^2 - 2\alpha K'U'(\Phi(K) - K') = c$$

or, with $\gamma = c/\alpha$

$$K'^2 - (\beta K - C^*)^2 = \gamma. \tag{51}$$

This equation can be solved for K' as soon as the arbitrary constant on the right-hand side can be assigned a specific value. Note that this constant is to hold for all t including $t \to \infty$. The convergence of the improper integral (49) implies that: $\beta K - K' - C^*$ tend to zero, this means that $\gamma = 0$. The by integration $K(t) = K^* - Dexp(-\beta t)$ where $K^* = C^*/\beta$ and D is a constant. For fixed $K(t_1) = K_1$ we obtain:

$$K(t) = K^* - (k - k^*)exp^{(-\beta t - t_1)}, \quad t_1 \leq t < \infty$$

But if we consider only cases where the time interval is finite, γ is a positive constant, in the phase plane (K, K') the set of optimal solutions represent a family of hyperboles with center $k' = 0, K = K^* = C^*/\beta$ and asymptotes $K' = \pm(\beta K - C^*)$.

We take $y = K - C^*/\beta = K - K^*, \ \ \gamma = \beta^2 H^2$ and equation (51) becomes $y'^2 = \beta^2(H^2 + y^*)$ Hence $y(t) = -H \sinh \beta(t^* - t)$ and the arc of trajectory is given by:

$$K(t) = K^* - H \sinh \beta(t^* - t) \quad t_1 \leq t \leq t^*,$$

where $H = \gamma^{1/2}\beta^{-1}$ and t^*.

Exercise 11 *Discuss Ramsey's model with* $\Phi(K) = \beta K$ *and* $U(C) = -U_0 e^{-\alpha c}, \ \ \alpha$ *constant.*

15.4 A control formulation of a classical problem of variational calculus

In this subsection we will obtain the classical Euler equation (the first order condition) from an optimal formulation of a simple problem of the variational calculus.

The general simple problem in the calculus of variations $\mathcal{V}$ can be written as a control problem by relabelling $\dot{x}$ as u. (Recall that u denotes the control variable). Then this problem of the variational calculus becomes the following control problem. Minimize:

$$C(u) = \int_{t_0}^{t_1} L(t, x(t), u(t)(t))dt + l(x(t_0)) + l(x(t_1))$$

subject to the state equations:

$$\dot{x} = u(t)$$

The function H in the present case is given by the formula

$$H(t, x, u, \eta) = \eta^0 f^0(t, x, u) + \eta u.$$

Let x^* the solution of the variational problem, then $(x^*, u^*) = (x^*, \dot{x}^*)$ is a solution of the corresponding control problem.

Observe that the pair (x^*, u^*) satisfies the maximal principle: There exists $\bar{\eta}^*(t) = \{\eta^{0*}, \eta^*(t)\} \neq 0$ for all $t \in [t_0, t_1]$ and such that for a.e. $t \in [t_0, t_1]$

$$\begin{aligned} \dot{x}(t) = H_\eta(t, x^*, u^*, \bar{\eta}^*) = u^*(t) \quad (a) \\ \dot{\eta} = -H_x(t, x^*, u^*, \bar{\eta}) = -\eta^{0*}\tfrac{\partial f^0}{\partial x}(t, x^*, (t)u^*(t)) \quad (b) \end{aligned} \tag{52}$$

and $H(t, x, u, \eta) = \eta^0 f^0(t, x, u) + \eta u$ is maximized with respect to u at (x^*, u^*).

We assert that $\eta^0 \neq 0$, because in other case $\eta^*(t)u^*(t) \geq \eta^*(t)u$ for al $u \in \Omega$. It follows that in this case $\lambda(t) = 0$, which can not be. Since $\eta^0 \neq -1$.

Since the mapping $u \rightarrow \bar{H}(t, x, u, \bar{\eta})$ is differentiable it follows:

$$H_u(t, x^*(t), u^*(t), , -1, \bar{\eta}) = 0.$$

and therefore

$$\eta^*(t), = f^0(t, x^*(t), u^*(t),) \;\; \forall t \in [t_0, t_1].$$

From this equation and from the adjoint equation we obtain the Euler equation:

$$\frac{\partial f^0}{\partial u}(t, x^*(t), \dot{x}^*(t)) = \int_{t_0}^{t_1} \frac{\partial f^0}{\partial x}(s, x^*(s)), \dot{x}^*(s))ds$$

or

$$\frac{\partial f^0}{\partial x}(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}[\frac{\partial f^0}{\partial \dot{x}^*}(t, x^*(t), \dot{x}^*(t))].$$

For more details see [Berkovitz, L.D.].

15.5 Optimal control formulation of the Bolza problem

The problem of Bolza in the calculus of variations differs from the simple problem in that in addition to classical variational equations, new restraint conditions are required.

The problem to be considered is that of minimizing the function:

$$I_0 = g_0(b) + \int_{t_1}^{t_2} L_0(t, x(t), u(t), b)dt, \tag{53}$$

satisfying a system of differential equations:

$$\dot{x}_i(t) = f_i(t, x(t), u(t), b), \tag{54}$$

$$\begin{aligned} &\phi_\alpha(t, x(t), u(t), b) \leq 0, \quad 1 \leq \alpha \leq m', \\ &\phi_\alpha(t, x(t), u(t), b) = 0, \quad m' < \alpha \leq m, \end{aligned} \tag{55}$$

a set of initial and terminal conditions:

$$t_s = T(b^s), \quad x_i(t_s) = Xis(b) \tag{56}$$

and a set of isoperimetric relations:

$$\begin{array}{c} I_\gamma \leq 0 \\ I_\gamma = 0 \end{array} \tag{57}$$

where

$$I_\gamma = g_\gamma(b) = \int_{t_1}^{t_2} L_\gamma(t, x(t), u(t), b)dt. \tag{58}$$

We assume:

(A1) All functions are continuously differentiable on a set X of points in the $(x, u, b, t)-$space. The set X_0 of all elements in X satisfying (55) will be called the **set of admissible elements.**

(A2) The matrix $\left(\frac{\partial \phi}{\partial u}, \delta_{ij}\phi_j\right)$ has rank m at each element $(x(t), u(t), b, t) \in X_0$, where δ_{ij} is the Kroneker's delta, and $\partial\phi/\partial u$ is the jacobian matrix of ϕ evaluated at $(x(t), u(t), b, t)$

If these conditions are satisfied, a complicate dependence of the response $x(t)$ and the controllers $u(t)$ can be determined trough a dynamical differential equations, and the variational problem becomes a control problem. For details see [Berkovitz, L.D.].

16 The Hestenes' Theorem.

Suppose that the arc $x_0(t), u_0(t), b_0, \quad t_1 \leq t \leq t_2$ affords a minimun to I_0 Then there exist multipliers:

$$\lambda_0 \leq 0, \lambda_\gamma, p_i(t), \mu_\alpha(t), \gamma = 1, ..., p; i = 1, ..., n; \alpha = 1, ..., m;$$

not vanishing simultaneously on $t_1 \leq t \leq t_2$ and functions:

$$H(t, x, u, b, p, \mu) = p_i f_i - \lambda_0 L_0 - \lambda_\gamma L\gamma - \mu_\alpha \phi_\alpha$$

$$g(b) = \lambda_0 g_0 + \lambda_\gamma g_\gamma$$

such that the following relations hold:

- (1)The multipliers λ_γ are constant and $\lambda_\gamma \geq 0$ with $\lambda_\gamma = 0$ in case $I_\gamma < 0$.
- (2)The multipliers $\mu_\alpha(t)$ are piecewise continuous and are continuous at each point of continuity of $u_0(t)$. Moreover for each $\alpha \leq m'$ the relation $\mu_\alpha(t) \geq 0$ holds and the equation

$$\mu_\alpha(t)\phi_\alpha(t, x_0(t), u_0(t), b_0) = 0 \tag{59}$$

  holds on $t_1 \leq t \leq t_2$.
- (3) The multipliers $p_i(t)$ are continuous and have piecewise continuous derivatives. In fact there are constants, c_i, c such that the relations

$$p_i = -\int_{t_1}^{t_2} H_x ds + c_i, \quad H = -\int_{t_1}^{t_2} H_t ds + c, \quad H_{u_k} = 0, \tag{60}$$

  hold along x_0.
- (4) The transversality condition:

$$-\lambda_0 \frac{\partial G}{\partial b_j} - \left[H \frac{\partial T^s}{\partial b_j} + \sum_{i=1}^{n} p_i(T^s) \frac{\partial x_i^s}{\partial b_j} \right] - \int_{t_1}^{t_2} \frac{\partial H_i^s}{\partial b_j} dt = 0, \tag{61}$$

  Where $j = 1, 2, ..., \alpha, \; s = 1, 2$, these are identities in b, on x_0.
- (5) The inequality

$$H(t, x_0(t), u, b_0, p(t), 0) \leq H(t, x_0(t), u_0(t), b_0, p(t), 0), \tag{62}$$

  holds whenever $(t, x_0(t), u, b_0) \in X_0$.

Equation (60) are equivalent to the statements that p_i, H are continuous along x_0 and that on each sub arc on which $u_0(t)$ is continuous we have,

$$\frac{dp_i}{dt} = -H_{x_i}, \quad \frac{dH}{dt} = \frac{\partial h}{\partial t}.$$

In the inequality (62) we have set $\mu_\alpha = 0$. If one wish to retain $\mu_\alpha = \mu_\alpha(t)$ we shall write the inequality:

$$H(t, x_0(t), u, b_0, p(t), \mu) + \mu_\alpha \phi_\alpha(t, x_0(t), u, b_0) \leq$$
$$\leq H(t, x_0(t), u_0(t), b_0, p(t), \mu)$$

17 Some applications

In this section we consider three important applications of the control theory. By means of the first example we illustrate the Hestenes' theorem. We again consider the problem of optimal growth because familiarity with this subject will help the reader to understand the theory developed in the last section. The second example is an application of the optimal control to differential game theory, and the last is the well know *isoperimetric problem*

17.1 Optimal growth once again

We begin discussing again, the optimal growth problem with explicit consideration to inequality constraints.

Following the considerations that we did in 13.1, we can write the problem as follows:

Maximize the welfare function

$$\max_{c,i} \int_0^\infty u(c(t))e^{-\delta t} dt$$

subject to:

$$\begin{aligned} &\dot{k} = f(k(t)) - c(t) - \lambda k(t) && (\mathbf{a}) \\ &0 \leq f(k(t)) - c(t) - i(t) && (\mathbf{b}) \\ &k(0) = k_0, \;\; k(t) \geq 0, \;\; c(t) \geq 0, && (\mathbf{c}) \end{aligned} \tag{63}$$

where δ is a discount rate, and utility per unit of consumption is given by $u(c(t))$ $u'(c) > 0$, $u''(c) < 0$, represent positive but diminishing marginal utility (concavity of the utility function).

We first proceed without explicit consideration of the state variable constraint (63, c). Introducing the multipliers $\eta(t), r(t)$ and $\nu(t)$ we define the function L as follows:

$$L \equiv L[k(t), c(t), i(t), t, \eta(t), r(t), \nu(t)] = \tag{64}$$

$$u(c)t)e^{-\delta t} + \eta(t)(i(t) - \lambda k(t)) + r(t)(f(k(t)) - c(t) - i(t)) + \nu(t)c(t).$$

Note that the rank constraint qualification (condition A2 in 15.5)is trivially satisfied:

$$\begin{vmatrix} \frac{\partial}{\partial x}[f(k) - c(t) - i(t)] & \frac{\partial}{\partial i}[f(k) - c(t) - i(t)] \\ \frac{\partial x}{\partial x} & \frac{\partial i}{\partial x} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0.$$

Then in view of the Hestenss Theorem the solution (k^*, c^*, i^*) of the above problem must satisfy

- (i) the following conditions:

$$\dot{k}(t) = \frac{\partial L^*(t)}{\partial \eta(t)}\ , \ \ \dot{\eta}(t) = -\frac{\partial \bar{L}(t)}{\partial k(t)} \tag{65}$$

$$\frac{\partial L*(t)}{\partial c(t)} = 0 \qquad \frac{\partial L^*(t)}{\partial i(t)} = 0, \ \ where\ \ \bar{L} = L(k^*(t), c^*(t), i^*(t), t, \eta(t), r(t), \nu(t)). \tag{66}$$

- (ii) The relations:

$$\begin{aligned} & r(t) \geq 0 \quad r_t(f(k^*(t)) - c(t) - i(t)) \\ & \nu(t) \geq 0 \quad \nu(t)c(t) = 0. \end{aligned} \tag{67}$$

- (iii)

$$H(k^*(t), c^*(t), i^*(t), t, \eta(t)) \geq H(k^*(t), c(t), i(t)t, \eta(t))$$

  for all $k^*(t), c(t), i(t)$ which satisfy $f(k^*(t)) - c(t) - i(t) \geq 0$ and $c(t) \geq 0$, where

$$H(k(t), c(t), i(t)t, \eta t)) = u(c(t))e^{-\delta t} + \eta(t)(i_t - \lambda k(t))$$

- (iv) The right hand end-point condition

$$\lim_{t\to\infty} p(t) \geq 0, \ \ and \ \ \lim_{t\to\infty} p(t)k^*(t) = 0, \tag{68}$$

  must hold.

These condition can be rewritten

1. (65) as: $\dot{k}(t) = i(t) - \lambda k(t) \quad and \quad \dot{\eta}(t) = \lambda p(t) - r(t)f'(k^*(t))$
2. (66) as: $u'(c^*(t))e^{-\delta t} - r(t) + \nu(t) = 0, \ \ and\ \ \eta(t) - r(t) = 0.$

From item (3) and nonsatiation we obtain $\eta(t) \geq u'(c^*(t))e^{-\delta t} > 0$. Then $r(t) \geq 0$ follows and then $f(k^*(t)) - c^*(t) - i(t) = 0$. It is important to note that the equality constraint is obtained as a result of the nonsatiation assumption.

If we assume that $c(t) > 0$, then we must have $\nu(t) = 0$, so we obtain: $\eta e^{-\delta t} = u'(c(t)$.

The rest of the analysis is the same as that in (13.1).

17.2 Differential games and dynamic optimization. Pursuit Games

The most important class of two-person zero-sum differential games is the pursuit games, , in which player 1 is a pursuer and player 2 an evader. The games ends when the pursuer is sufficiently close to the evader, at which point the pursuer is said to *capture* the evader, the time to capture being the duration of the game. If the pursuer never comes sufficiently close to the evader to capture him, then the evader *escapes,* and the time to capture is infinite. This description of the pursuit game is general enough to cover diverse situations: pursuit of the runner in a football game or the pursuit of a missile by an antimissile.

The simplest case is that of pursuit in the plane. The players are located at two points in the plane and move at fixed velocities, the velocity of the pursuer exceeding the velocity of the evader. The control variables are the directions in which the players move.

Line L is the reference direction, and line M passes trough the coordinates of both players at any time. The state variable are chosen as those in the moving reference system:

$x_1 =$ distance between player 1 and player 2.

$x_2 =$ angle between L and M.

The control variables are the directions of movement:

$u_1 =$ angle between velocity vector of player 1 and L.

$u_2 =$ angle between velocity vector of player 2 and L.

where player 1 (pursuer) moves with speed s_1, player 2 (evader) moves with speed s_2 $(s_1 > s_2)$ and: $0 \leq u_1 < 2\pi,\ \ 0 \leq u_2 < 2\pi$.

The equations of motion are:

$$\begin{aligned} \dot{x}_1 &= -s_1 \cos(u_1 - x_2) + s_1 \cos(u_2 - x_2) \\ \dot{x}_2 &= \tfrac{-s_1 \sin(u_1 - x_2) + s_1 \sin(u_2 - x_2)}{x_1} \end{aligned} \tag{69}$$

Terminal time t_1 is free, it is the time at which the distance between the players is reduced to a given distance $\mathcal{D}$, such that $x(t_1) = \mathcal{D}$ *at which time the pursuer captures the evader.*

The payoff of the pursuer is

$$J = -\int_{t_0}^{t_1} dt = -(t_1 - t_2).$$

The Hamiltonian is therefore:

$$-1+\eta_1(-s_1\cos(u_1-x_2)+s_1\cos(u_2-x_2))+\eta_2\frac{-s_1\sin(u_1-x_2)+s_1\sin(u_2-x_2)}{x_1}. \tag{70}$$

By the PMP, the Hamiltonian should be maximized with respect to u^1 and minimized with respect to u_2. The first order conditions are:

$$\begin{aligned}\tfrac{\partial H}{\partial u_1} &= \eta_1 s_1\sin(u_1-x_2)-\tfrac{\eta_2}{x_1}s_1\cos(u_1-x_2)=0\\ \tfrac{\partial H}{\partial u_2} &= \eta_1 s_2\sin(u_2-x_2)-\tfrac{\eta_2}{x_1}s_2\cos(u_2-x_2)=0\end{aligned} \tag{71}$$

Implying:

$$\tan(u_1-x_2)=\tan(u_2-x_2)=\frac{\eta_2}{\eta_1 x_1}. \tag{72}$$

The adjoint system is:

$$\begin{aligned}\dot{\eta}_1=-\tfrac{\partial H}{\partial x_1} &= -\tfrac{\eta_2}{x_1^2}\left(-s_1\sin(u_1-x_2)+s_2\sin(u_2-x_2)\right)\\ \dot{\eta}_2=-\tfrac{\partial H}{\partial x_2} &= -\eta_1\left(-s_1\sin(u_1-x_2)+s_2\sin(u_2-x_2)\right)+\\ &\quad \tfrac{\eta_2}{x_1}\left(s_1\sin(u_1-x_2)-s_2\cos(u_2-x_2)\right).\end{aligned} \tag{73}$$

From (72) y (73) it follows $\dot{\eta}_2=0$; i:e y_2 is constant through time. Also as there is no constraint on the terminal value x_2 we obtain that $\eta_2(t_1)=0$ and then $\eta_2(t)=0 \;\; t_0\le t\le t_1$.

So substituting in (71) it follows that:

$$u_1=x_1 \quad \text{and} \quad u_2=x_2.$$

This is the case in which the pursuer moves directly toward the evader, ,and the evader directly away from the pursuer. In this case the distance between the players follows from the differential equation $\dot{x}_1=s_2-s_1$ so:

$$x(t)=(s_1-s_2)(t_0-t)+x(t_0).$$

By definition of t_1 :

$$t_1=t_0-\left(\frac{l-x_1(t_0)}{s_1-s_2}\right).$$

And the value of the game is:

$$J^*=-(t_1-t)=\left(\frac{l-x_1(t_0)}{s_1-s_2}\right).$$

If for instance, the evader moves away from the pursuer forming a (non-null) angle with the direction of straight line M, the pursuer catches the evader in a shorter time.

17.3 The Isoperimetric problem from a Control Theory point of view

Suppose that we impose the following additional constraint on admissible pairs (u, x)

$$\int_{t_0}^{t_1} h_i(t, x(t), u(t))dt \overset{=}{} c_i \quad i = 1, ..., r$$

where $h = (h_1, ..., h_r)$ as the same properties as f (53), and $c = (c_1, ..., c_r)$ is a given vector. Let

$$\bar{H}(t, \bar{x}, u, \bar{\eta}, \lambda) = \bar{H}(t, \bar{x}, u, \bar{\eta}) + \lambda h(x, t, u).$$

Exercise 12 *Show that if $g \equiv 0$ and if (u^*, x^*) is an optimal pair in the interval $[t_0, t_1]$, for the isoperimetric problem just formulated, then the following maximal principle holds:*

There exist a constant $\eta_0^* \leq 0$, an absolutely continuous vector functions $\eta^* = (\eta_1^*(t), ..., \eta_n^*(t))$ and a constant vector $\lambda^* \in R^r$ such that the following hold:

1. *The vector $(\eta_0^*, \eta^*, \lambda^*)$ is never zero on $[t_0, t_1]$.*
2. *For a.e t on $[t_0, t_1]$,*
$$\dot{x}(t) = \frac{\partial}{\partial \eta} \bar{H}(t, \bar{x}^*, u^*, \bar{\eta}^*, \lambda^*)$$
$$\dot{\eta}^*(t) = -\frac{\partial}{\partial x} \bar{H}(t, \bar{x}^*, u^*, \bar{\eta}^*, \lambda^*).$$
3. *For any admissible u defined on $[t_0, t_1]$,*
$$\int_{t_0}^{t_1} \bar{H}(t, \bar{x}^*(t), u^*(t), \bar{\eta}^*(t), \lambda^*)dt \geq \int_{t_0}^{t_1} \bar{H}(t, \bar{x}(t), u(t), \bar{\eta}(t), \lambda^*)dt$$
4. *If $t \to (\bar{f}(t, x(t), u(t)), h(t, x(t), u(t))$ is continuous then the following transversality conditions are satisfied:*
$$(\bar{H}(t_i, \bar{x}(t_i), u^*(t_i), \bar{\eta}^*(t_i), \lambda), -\bar{\eta}(t_i))$$
is orthogonal to the tangent hyperplane π_i at $X_i, i = 0, 1$.

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