Universidad de la República
Facultad de Ciencias Sociales
DEPARTAMENTO DE ECONOMIA

Documentos de trabajo

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Documento No. 05/03
Setiembre, 2003

Comparative Statics, English Auctions, and the Stolper-Samuelson Theorem *

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September 21, 2003

Abstract

Changes in the parameters of an n-dimensional system of equations induce changes in its solutions. For a class of such systems, we determine the qualitative change in solutions given certain qualitative changes in parameters. Our methods and results are elementary yet useful. They highlight the existence of a common thread, our "own effect" assumption, in formally diverse areas of economics. We discuss several applications; among them, we establish the existence of efficient equilibria in English auctions with interdependent valuations, and a version of the Stolper-Samuelson Theorem for an $n \times n$ trade model.

Resumen. Los cambios en parámetros de un sistema de n ecuaciones inducen cambios en sus soluciones. Para una clase de sistemas, determinamos cómo ciertos cambios cualitativos en los parámetros inducen ciertos cambios cualitativos en sus soluciones. Nuestros métodos y resultados son elementales pero útiles. Muestran la existencia de un hilo conductor, nuestra condición de "efecto propio", en áreas formalmente muy diversas de economía. Discutimos varias aplicaciones, entre ellas establecemos la existencia de un equilibrio eficiente en subastas inglesas con valuaciones interdependientes y una versión del Teorema de Stolper y Samuelson para un modelo $n \times n$ de comercio internacional.

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*JEL Classification: C60, D44, F11. We thank Chris Shannon for comments on a very early version of these results.

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1 Introduction

Consider a model with n exogenous variables, $\mathbf{p} = (p_1, \ldots p_n)$, and n endogenous variables $\mathbf{x} = (x_1, \ldots x_n)$ that are determined from the equation

$$\mathbf{p} = \mathbf{v}(\mathbf{x}). \tag{1}$$

The function $\mathbf{v}$ is a primitive of the model. Say that $\mathbf{x}(\mathbf{p})$ is determined from equation 1. *We want to know how* $\mathbf{x}(\mathbf{p})$ *varies with* $\mathbf{p}$. This structure is ubiquitous in economics.

We assume that the function $\mathbf{v}$ satisfies a simple condition relating changes in the coordinates of $\mathbf{x}$ to changes in the coordinates of $\mathbf{v}(\mathbf{x})$. Roughly, we assume that changes in x_i are more important for changes in v_i than changes in x_h, for $h \neq i$. We call the effect of x_i on v_i an "own effect;" we call our property the "own-effect property." The own-effect property can be interpreted as a single-crossing condition in the context of auctions, and as a factor-intensity condition in the context of trade models.

We prove various facts about $\mathbf{x}(\mathbf{p})$, among them that, provided $\mathbf{v}$ is monotone increasing and satisfies the own-effect property, certain changes in $\mathbf{p}$ make certain components of $\mathbf{x}(\mathbf{p})$ increase and others decrease. These facts imply some important results in very different areas of economics.

Our results and their proofs are elementary. They are also powerful. We use them to simplify and generalize two important theorems in very different areas of economics: Maskin's theorem on the existence of an efficient equilibrium in English auctions with interdependent valuations, and the Stolper-Samuelson Theorem of trade theory.

In the remainder of the Introduction, we review briefly the usual methods used to determine how $\mathbf{x}$ varies with $\mathbf{p}$, and we discuss applications.

Consider then the question of how $\mathbf{x}$ varies with $\mathbf{p}$. If a local answer suffices, the Implicit Function Theorem—which involves assuming that $\mathbf{v}$ is C^1, that the solutions to the equation $\mathbf{p} = \mathbf{v}(\mathbf{x})$ are interior, and that $\mathbf{v}$'s Jacobian matrix is non-singular at a solution—provides an answer, and also establishes that the solution to $\mathbf{v}(\mathbf{x}) = \mathbf{p}$ is locally unique.

If a global answer is desired, the Gale-Nikaido Theorem (Gale and Nikaido (1965)) is the proper tool. It states that, if $\mathbf{v}$ is continuously differentiable, and the Jacobian of $\mathbf{v}$ is everywhere a P-matrix—all the principal minors of $\mathbf{v}$ are positive—then $\mathbf{v}$ is globally invertible. The solution to $\mathbf{v}(\mathbf{x}) = \mathbf{p}$ is in this case unique.

Our approach does not require differentiability, and does not yield the uniqueness or even the existence of a solution, issues that must be addressed separately in applications. But besides making a technical point—we can generalize certain results—our approach is useful because it shows that the simple, and economically intuitive, notion of an "own effect" is behind results that are formally very diverse.

We now discuss applications: we use our results to obtain new results in auction theory and trade theory. We also derive a simple application to the comparative statics of factor demands.

We study single-object English auctions with potentially asymmetric bidders—i.e., bidders need not be ex ante identical—and with interdependent valuations. Each bidder observes a random signal and each bidder's valuation depends on the realization of the entire signal-profile. Our own-effect property restricts the possible changes in valuations for certain changes in signals. It generalizes to n bidders the Single Crossing Property first assumed by Maskin (1992) in a model of 2 bidders. We establish the existence of an efficient ex post equilibrium with n bidders.

The Stolper-Samuelson Theorem of trade theory says that, if there are two consumption-goods and two production-factors, and the production of good 1 is relatively more intense in the use of factor 1, then an exogenous increase in the price of good 1 will bring about an increase in the price of factor 1 and a decrease in the price of factor 2.

In the context of a trade model, our own-effect property generalizes the notion of "relatively more intense" from the Stolper-Samuelson Theorem. Our results imply that the conclusion of the Stolper-Samuelson Theorem is true with more than two goods and factors, and under quite general conditions on production functions.

In Section 2 we present our main results. In Section 3 we present our results on English auctions, and in Section 4 we give a version of the Stolper-Samuelson Theorem. In Section 5 we present an application to the comparative statics of factor demands.

2 Some Global Comparative Statics

2.1 Notation

Let $n \geq 2$ be a natural number and $\mathbf{v} : I\!R^n \to I\!R^n$ a function, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Elements of the domain of $\mathbf{v}$ are typically denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, elements of the image of $\mathbf{v}$ are typically denoted by $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

For any x, x' in $I\!R^n$, we say that $x \leq x'$ if $x_i \leq x'_i$ for all i, that $x < x'$ if $x \leq x'$ and $x \neq x'$, and that $x \ll x'$ if $x_i < x'_i$ for all i. A function $\mathbf{v}$ is said to be monotone nondecreasing if $\mathbf{x} \leq \mathbf{x}'$ implies $\mathbf{v}(\mathbf{x}) \leq \mathbf{v}(\mathbf{x}')$, and it is monotone increasing if $\mathbf{x} < \mathbf{x}'$ implies $\mathbf{v}(\mathbf{x}) \ll \mathbf{v}(\mathbf{x}')$.

2.2 The own-effect property

If no assumptions are made about the function $\mathbf{v}$, very little can be said about how the solution to $\mathbf{v}(\mathbf{x}) = \mathbf{p}$ varies with $\mathbf{p}$. We place a restriction on the relative effect that a change from $\mathbf{x}$ to $\mathbf{x}'$ has on the coordinates of $\mathbf{v}$: coordinates h for which $x'_h < x_h$ must experience changes in v_h that are dominated by the change in v_i for some coordinate i for which $x'_i \geq x_i$. We make this intuitive description precise with a definition.

Definition 1. *The function $\mathbf{v}$ satisfies the own-effect property (OEP) if, for any $\mathbf{x}$ and $\mathbf{x}'$ with $\mathbf{x}' \not\leq \mathbf{x}$ and $\mathbf{v}(\mathbf{x}') \not\leq \mathbf{v}(\mathbf{x})$, there is i such that*

$$v_i(\mathbf{x}') - v_i(\mathbf{x}) > v_h(\mathbf{x}') - v_h(\mathbf{x})$$

for all h with $x'_h \leq x_h$.

Remark 1. Note that $x'_i > x_i$ for the i identified by the definition.

For example, consider the case of $n = 2$, represented in Figure 1. Suppose $\mathbf{v}$ satisfies the OEP, and pick $\mathbf{x}$ and $\mathbf{x}'$ such that $x'_1 - x_1 > 0$ and $x'_2 - x_2 < 0$, so $\mathbf{x}' - \mathbf{x}$ is in orthant **4** in Figure 1. The OEP only has bite if $\mathbf{v}(\mathbf{x}') \not\leq \mathbf{v}(\mathbf{x})$; so assume that $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ is not in orthant **3**.

The OEP says that the increase in x_1 has a larger effect on v_1 than on v_2, and that the decrease in x_2 has a larger effect on v_2 than on v_1. For example the OEP is satisfied if $v_1(\mathbf{x}') > v_1(\mathbf{x})$ and $v_2(\mathbf{x}') < v_2(\mathbf{x})$, so $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ is also in orthant **4**. More generally, the OEP requires that $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ lie below the $x_2 = x_1$ line. In Figure 1, $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ must lie in the area marked

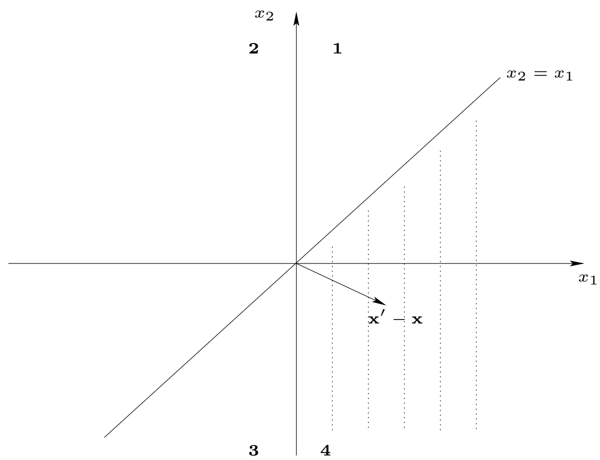

Figure 1: The OEP

with dotted lines. Similarly, if $x'_1 - x_1 < 0$ and $x'_2 - x_2 > 0$, then the OEP requires that $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ lie in orthants **1** or **2**, above the $x_2 = x_1$ line.

Theorem 2, and its corollaries, relate the location of $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ to the location of $\mathbf{x}' - \mathbf{x}$. If $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ lies in orthant **2**, and $\mathbf{v}$ satisfies the OEP, then $\mathbf{x}' - \mathbf{x}$ cannot lie in orthant **4**. Then, either $\mathbf{x}' \leq \mathbf{x}$ or $x'_2 > x_2$—so $v_2(\mathbf{x}') > v_2(\mathbf{x})$ implies that $x'_2 > x_2$.

If, in addition, we assume that $\mathbf{v}$ is monotone increasing we can rule out that $\mathbf{x}' - \mathbf{x}$ lies in orthants **1** or **3** as well. Then it must be that $\mathbf{x}' - \mathbf{x}$ lies in orthant **2**, so we get $x'_1 < x_1$ in addition to $x'_2 > x_2$.

In sum, if $p'_1 = v_1(\mathbf{x}') > p_1 = v_1(\mathbf{x})$ and $p'_2 = v_2(\mathbf{x}') < p_2 = v_2(\mathbf{x})$ then it must be that $x'_1 > x_1$ and $x'_2 < x_2$. Thus the OEP determines—at least qualitatively—how x depends on p.

A similar exercise can be carried out with any $n > 2$. In $I\!R^n$, the *OEP* implies that $\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})$ must be in a convex cone that is completely determined by the orthant containing $\mathbf{x}' - \mathbf{x}$.

2.3 Results

In order to state the theorem, we identify first the coordinates in which $\mathbf{v}(\mathbf{x}')$ dominates $\mathbf{v}(\mathbf{x})$. Given $\mathbf{v}$, $\mathbf{x}$ and $\mathbf{x}'$, denote by $J \subseteq \{1, \ldots n\}$ the set of indexes i such that $v_i(\mathbf{x}') - v_i(\mathbf{x}) > 0$.

Theorem 2. *Let* $\mathbf{v}$ *satisfy the OEP. If* $\mathbf{x}' \nleq \mathbf{x}$*,* $J \neq \emptyset$*, and* $v_i(\mathbf{x}') - v_i(\mathbf{x}) = v_j(\mathbf{x}') - v_j(\mathbf{x})$ *for all* $i, j \in J$*, then* $x'_j > x_j$ *for all* $j \in J$*. Further, if* $\mathbf{v}$ *is monotone increasing, then* $x'_h < x_h$ *for some* $h \notin J$*.*

Proof. Let $j \in J$. By the OEP there is i such that $v_i(\mathbf{x}') - v_i(\mathbf{x}) > v_h(\mathbf{x}') - v_h(\mathbf{x})$ for all h with $x'_h \leq x_h$. If $i \in J$ then $v_i(\mathbf{x}') - v_i(\mathbf{x}) = v_j(\mathbf{x}') - v_j(\mathbf{x})$; so $x'_j > x_j$. If $i \notin J$ then $v_i(\mathbf{x}') - v_i(\mathbf{x}) < v_j(\mathbf{x}') - v_j(\mathbf{x})$; so $x'_j > x_j$.

Further, if $\mathbf{v}$ is increasing, $\mathbf{x} < \mathbf{x}'$ implies $\mathbf{v}_h(\mathbf{x}') < \mathbf{v}_h(\mathbf{x})$ for all h. So we must have $x'_h < x_h$ for some $x \notin J$. □

Remark 3. If $\mathbf{v}$ is monotone nondecreasing, $J \neq \emptyset$ implies $\mathbf{x}' \nleq \mathbf{x}$, which is useful for interpreting Theorem 2.

Corollaries 4 and 5 are simple consequences of Theorem 2, but they are useful in the applications we develop below.

Corollary 4. *Let* $\mathbf{v}$ *be monotone increasing and satisfy the OEP. Let* $p_i = v_i(x)$ *and* $p'_i = v_i(x')$ *for all* i*. If, for some* j*,* $p'_j > p_j$ *and* $p'_h \leq p_h$ *for all* $h \neq j$*, then* $x'_j > x_j$ *and* $x'_h < x_h$ *for at least one* $h \neq j$*.*

Proof. First, $\mathbf{x}' \nleq \mathbf{x}$, as $\mathbf{v}$ is monotone increasing. Second, $J = \{j\}$ so Theorem 2 implies that $x'_j > x_j$ and that there is h such that $x'_h < x_h$ □

Corollary 5. *Let* $\mathbf{v}$ *be monotone increasing and satisfy the OEP. Let* $\mathbf{x}$ *and* $\mathbf{x}'$ *be such that* $p = v_i(x)$ *and* $p' = v_i(x')$ *for all* i*. If* $p < p'$ *then* $x_i < x'_i$ *for all* i*.*

Proof. First, $\mathbf{x}' \nleq \mathbf{x}$, as $\mathbf{v}$ is monotone nondecreasing. Second, $p < p'$ implies that $J = \{1, \ldots n\}$. By Theorem 2, $x'_i > x_i$ for all i. □

3 English Auction With Interdependent Valuations

In symmetric environments, English auctions of a single object have several desirable properties. When bidders' valuations are private information—i.e., each bidder's valuation for the object is not affected by the information that other bidders possess—English auctions, at least in the clock format, implement sincere bidding in dominant strategies, generate efficient outcomes, and with adequate reserve prices, maximize the seller's expected revenue. When bidders' valuations have a common-value component—i.e, a bidder's valuation for the object is affected by the private information of other bidders—English auctions have efficient outcomes, and generate higher expected revenue than the sealed-bid second-price or first-price auctions.[1]

Given the desirable characteristics of the English auction, it seems reasonable to inquire when these mechanisms have efficient equilibria in asymmetric environments with interdependent valuations. In this we follow Maskin (1992) and Krishna (2001).

In the model each bidder observes a signal. For each bidder there is a distinct function that determines the bidder's valuation given the signal profile. Maskin (1992) shows in a two-bidder model that a single crossing property (SCP)—i.e., that bidder i's own signal has a larger influence on bidder i's value than on any other bidder—suffices to prove both that the English auction has a Nash equilibrium and that this equilibrium is efficient. Indeed existence and efficiency are joint products of Maskin's argument.

Our OEP is a generalization of the Maskin's SCP to n bidders, and allows to establish the existence of an ex post equilibrium of the English auction.

The generalization is not trivial; Krishna (2001) includes a three-bidder example satisfying Maskin's SCP (applied pairwise), but where the English auction does not have an efficient equilibrium. (Krishna attributes the idea of the example to Phil Reny.)

Our OEP is not the only generalization of Maskin's SCP. Krishna (2001) offers two other alternatives, his average crossing condition and his cyclical crossing condition. Both conditions are local and once again, existence and efficiency are obtained as by-products of the same argument.

Our OEP (as well as Krishna's conditions) are satisfied in various useful and widely applied models; such as Wilson's (1998) log-normal model. The OEP is a global property and not immediately comparable with Krishna's local conditions. Differentiability of the value functions is not necessary in our setting. The OEP is relatively simple to verify in applications.

We believe—but we are happy to stand corrected—that Krishna's conditions, although local, would be difficult to verify unless they hold globally.

[1]This is due to the linkage effect (Milgrom and Weber (1982)). See, for instance, McAfee and McMillan (1987) and their references for precise statements of results and assumptions.

3.1 Model and Theorem

There are n bidders, indexed by $i = 1, \dots n$. Each bidder i receives a signal x_i about the value of the object being auctioned. The vector of signals $\mathbf{x} = (x_1, \dots x_n)$ is drawn at random from the set $[0, \omega]^n \subseteq I\!R^n$. If a vector $\mathbf{x}$ of signals is realized, i's valuation for the object is $v_i(\mathbf{x})$. We assume that $v_i(\mathbf{0}) = 0$, and that v_i is monotone non-decreasing. Denote by $\mathbf{v} = (v_i)_{i=1}^n$ the collection of such functions.

We will prove that provided $\mathbf{v}$ satisfies the OEP, the English auction has an efficient ex-post equilibrium.

The following example identifies one class of valuation functions $\mathbf{v}$ that satisfy the OEP.

Example Let $u_i : [0, \omega] \to I\!R$ be a monotone increasing function, for all i. Let $w : [0, \omega]^n \to I\!R$ be arbitrary. If $v_i(\mathbf{x}) = u_i(x_i) + w(\mathbf{x})$, then $\mathbf{v}$ satisfies the OEP.

To see this, let $x'_i > x_i$ and and $x'_h \leq x_h$. Then,

$$\begin{aligned} v_i(\mathbf{x}') - v_i(\mathbf{x}) &= u_i(x'_i) - u_i(x_i) + w(\mathbf{x}') - w(\mathbf{x}) \\ &> u_h(x'_h) - u_h(x_h) + w(\mathbf{x}') - w(\mathbf{x}) \\ &= v_h(\mathbf{x}') - v_h(\mathbf{x}), \end{aligned}$$

as $u_i(x'_i) - u_i(x_i) > 0$ and $u_h(x'_h) - u_i(x_h) \leq 0$.

We now show that with only two bidders Maskin's single crossing property implies the OEP.

Definition 2. *A* $\mathbf{v}$ *that is* C^1 *satisfies the global Maskin-SCP if, for all* i *and* h *with* $i \neq h$,

$$\frac{\partial v_i}{\partial x_i}(\mathbf{x}) > \frac{\partial v_h}{\partial x_i}(\mathbf{x}).$$

Proposition 6. *Let* $n = 2$. *If* $\mathbf{v}$ *is* C^1 *and satisfies the global Maskin-SCP, then it satisfies the OEP.*

Proof. Without loss of generality, let $x_1 > x_2$ and $x'_2 \leq x_2$. Then, $[v_1(\mathbf{x}') - v_1(\mathbf{x})] - [v_2(\mathbf{x}') - v_2(\mathbf{x})] =$

$$\int_{x_1}^{x'_1} [\frac{\partial v_1}{\partial x_1}(s, x_2) - \frac{\partial v_2}{\partial x_1}(s, x_2)] ds + \int_{x_2}^{x'_2} [\frac{\partial v_2}{\partial x_2}(x'_1, s) - \frac{\partial v_1}{\partial x_2}(x'_1, s)] ds > 0.$$

□

Maskin's proof of the existence of an ex post equilibrium with two bidders could be informally described as follows. Consider the system of equations

$$\begin{aligned} v_1(x_1, x_2) &= p \\ v_2(x_1, x_2) &= p. \end{aligned} \tag{2}$$

For a given price p, the indifference curves of both functions intersect on a unique signal vector (x_1, x_2), the single crossing property. That signal vector is a solution to the system of equations. As the price p increases, the solution (x_1, x_2) also increases in both coordinates. The implicit maps $x_i \mapsto p$ constitute an efficient ex post equilibrium of the English auction. (See Krishna (2001) for a full discussion.)

Krishna (2001) generalizes Maskin's argument showing that under his conditions, the average crossing or the cyclical crossing condition, the corresponding system of equations (2) with n bidders, has a monotone increasing solution, and that this implies that the English auction has an ex post equilibrium. To establish existence of a monotone increasing solution, Krishna differentiates the system (2) to obtain an equivalent system of differential equations. The Cauchy-Peano Theorem yields existence of a solution for each price p; Krishna's conditions imply that the solution is strictly monotone.

In order to establish our main result of the section, that if $\mathbf{v}$ satisfies the OEP then the English auction has an efficient ex-post equilibrium, we will also solve the system of equations $\mathbf{v}(\mathbf{x}) = p\mathbf{1}$.

The set $\mathbf{v}^{-1}(\mathbf{p1}) = \{\mathbf{x}|\mathbf{v}(\mathbf{x}) = p\mathbf{1}\}$ represents the intersection of the indifference curves with value p, one for each agent. The intersection need not be a singleton and it may even be empty. Corollary 5 states that, if the intersection is non-empty, any selection from $\{\mathbf{v}^{-1}(p\mathbf{1})$ is monotone; this fact will later be used to show that indeed such a selection is an equilibrium. That such a selection is non-empty, however, needs to be demonstrated. Lemma 8 states that under an additional boundary condition on $\mathbf{v}$ a solution to $\mathbf{v}(\mathbf{x}) = p\mathbf{1}$ exists. The proof of Lemma 8 uses Brower's Fixed Point Theorem. Finally Lemma 10 shows that when using the strategies obtained from Lemmas 5 and 8, the outcome is efficient.

Definition 3. $\mathbf{v}$ *is boundary constant if, for any i, and any $\mathbf{x}_{-i} \in [0, \omega]^{n-1}$, $v_i(\omega, \mathbf{x}_{-i}) = v_i(\omega\mathbf{1})$, and $v_i(0, \mathbf{x}_{-i}) = v_i(\mathbf{0})$.*

Before proceeding to state and prove Lemma 8, we argue with the following Proposition that the boundary condition is easily satisfied.

Proposition 7. *For any $\delta > 0$, there is a continuous, monotone non-decreasing, and boundary constant $\tilde{\mathbf{v}} : [0,\omega]^n \to I\!R^n$ such that $\tilde{\mathbf{v}}(\mathbf{0}) = \mathbf{0}$, $\tilde{\mathbf{v}}(\omega\mathbf{1}) = \mathbf{v}(\omega\mathbf{1})$, and $\tilde{\mathbf{v}}$ coincides with $\mathbf{v}$ on $[\delta, \omega-\delta]^n$.*

Proof. Let $\tilde{v_i}(\mathbf{x})$ equal $v_i(\mathbf{x})$ if $x_i \in (\delta, \omega-\delta)$; let $\tilde{v_i}(\mathbf{x})$ equal $v_i(\delta, \mathbf{x}_{-i})(x_i/\delta)$ if $x_i \le \delta$; let

$$\tilde{v_i}(\mathbf{x}) = \left[v_i(\omega\mathbf{1}) - v_i(\omega-\delta, \mathbf{x}_{-i})\right](x_i - \omega + \delta)/\delta + v_i(\omega-\delta, \mathbf{x}_{-i})$$

if $x_i \ge \omega - \delta$. Note that $\tilde{v_i}$ is continuous and monotone increasing. □

We are now ready to state our existence lemma.

Lemma 8. *If $\mathbf{v}$ is boundary constant, and $p \in (0, \min\{v_i(\omega\mathbf{1}) : i = 1, \ldots n\})$, then there is $\mathbf{x} \in [0,\omega]^n$ such that $p\mathbf{1} = \mathbf{v}(\mathbf{x})$.*

Proof. Let $g : I\!R \to (-1,1)$ be a continuous, strictly monotonically decreasing function such that $g(0) = 0$ (e.g. $g = 1/2 - \Phi$, where Φ is the Gaussian distribution function).

Let $\hat{v}_i(\mathbf{x}) = v_i(\mathbf{x}) - p$. Let $\mathbf{h} : [0,\omega]^n \to [0,\omega]^n$ be

$$h_i(\mathbf{x}) = \begin{cases} x_i + g\left(\hat{v}_i(\mathbf{x})\right) x_i & \text{if } \hat{v}_i(\mathbf{x}) > 0 \\ 0 & \text{if } \hat{v}_i(\mathbf{x}) = 0 \\ x_i + g\left(\hat{v}_i(\mathbf{x})\right)(\omega - x_i) & \text{if } \hat{v}_i(\mathbf{x}) < 0 \end{cases}$$

We shall verify that $\mathbf{h}$ satisfies the hypothesis of Brower's Fixed-Point Theorem. We begin by showing that $h_i(\mathbf{x}) \in [0,\omega]$. If $\mathbf{x}$ is such that $\hat{v}_i(\mathbf{x}) > 0$, then $-1 < g\left(\hat{v}_i(\mathbf{x})\right) < 0$. So $0 < h_i(\mathbf{x}) < x_i \le \omega$. If $\mathbf{x}$ is such that $\hat{v}_i(\mathbf{x}) \le 0$, then $1 \ge (\hat{v}_i(\mathbf{x})) \ge 0$. So $0 \le h_i(\mathbf{x}) \le x_i + (\omega - x_i) = \omega$.

It is easy, but tedious, to verify that $\mathbf{h}$ is continuous. Note that the only problematic point is $\mathbf{x}'$ such that $g\left(\hat{v}_i(\mathbf{x}')\right) = 0$; but $\lim_{\mathbf{x}\to\mathbf{x}'} h_i(\mathbf{x}) = x_i'$, and $h_i(\mathbf{x}') = x_i'$.

By Brower's Fixed-Point Theorem, there is $\mathbf{x}^* \in [0,\omega]^n$ such that $\mathbf{x}^* = \mathbf{h}(\mathbf{x}^*)$. First we shall prove that $\mathbf{x}^* \in (0,\omega)^n$. Suppose, by way of contradiction, that $x_i^* \in \{0,\omega\}$ for some i. If $x_i^* = \omega$, then $v_i(\omega\mathbf{1}) = v_i(\mathbf{x}^*)$ because $\mathbf{v}$ is boundary constant. Then $v_i(\mathbf{x}^*) > p$, so $\hat{v}_i(\mathbf{x}^*) > 0$ and $h_i(\mathbf{x}^*) = [1 - |g\left(\hat{v}_i(\mathbf{x})\right)|]\omega$ and thus $h_i(\mathbf{x}^*) < \omega$. This is impossible since $h_i(\mathbf{x}^*) = \omega$. If $x_i^* = 0$, then $0 = v_i(0) = v_i(\mathbf{x}^*)$ because $\mathbf{v}$ is boundary constant. Then $v_i(\mathbf{x}^*) < p$, so $\hat{v}_i(\mathbf{x}^*) < 0$ and $h_i(\mathbf{x}^*) = g\left(\hat{v}_i(\mathbf{x})\right)$. This is impossible since $0 = x_i^* = h_i(\mathbf{x}^*)$.

Second, we prove that $\mathbf{v}(\mathbf{x}^*) = p\mathbf{1}$. Fix an i. The equation $h_i(\mathbf{x}^*) = x_i^*$ implies that either $g\left(\hat{v}_i(\mathbf{x}*)\right) x_i = 0 = g\left(\hat{v}_i(\mathbf{x})\right)(\omega - x_i)$ or that $\hat{v}_i(\mathbf{x}^*) = 0$. Since $x_i^* \in (0, \omega)$ and $g \in (-1, 1)$, we conclude that $\hat{v}_i(\mathbf{x}^*) = 0$, and thus $v_i(\mathbf{x}^*) = p$. □

Because of Lemma 8, the functions σ_i below are well defined; they map each price p to a signal profile in the intersection of the indifference curves, i.e. a solution to the system of equations corresponding to (2). The inverse of these functions are the basis of the equilibrium bidding strategies.

Definition 4. *For each $p \in (0, \min\{v_i(\omega\mathbf{1}) : i = 1, \ldots n\})$, let $\sigma(p) = (\sigma_i(p))_{i=1}^n$ such that $p = v_i(\sigma(p))$ for all i. Extend σ to $[0, \min\{v_i(\omega\mathbf{1}) : i = 1, \ldots n\}]$ by $\sigma_i(0) = \lim_{p\to 0} \sigma_i(p)$, and similarly for $\sigma_i(\omega)$.*

Remark 9. The function σ is continuous.

Lemma 10 states that provided bidders use the strategies implicit in Lemma 8 and Definition 4, the outcome is efficient.

Lemma 10. *Let $\mathbf{v}$ satisfy the OEP. If $\mathbf{p}$ is such that $p_j > p_n$ for all $j \neq n$, and $\mathbf{x} = \sigma(\mathbf{p})$, then there is i such that $v_i(\mathbf{x}) > v_n(\mathbf{x})$.*

Proof. Let $j \neq n$. Since σ_j is strictly increasing, $x_j = \sigma_j(p_j) > \sigma_j(p_n)$. Then $\mathbf{x}_j > \sigma_j(p_n))\forall j \neq n$ and $\mathbf{x}_n > \sigma_n(p_n))$. By the OEP, there is i such that

$$v_i(\mathbf{x}) - v_i(\sigma(p_n)) > v_n(\mathbf{x}) - v_n(\sigma(p_n)).$$

But $v_i(\sigma(p_n)) = p_n$ and $v_n(\sigma(p_n)) = p_n$, so $v_i(\mathbf{x}) > v_n(\mathbf{x})$. □

We now state the main result of the section.

Theorem 11. *If $\mathbf{v}$ satisfies the OEP and is boundary constant, then there is an efficient (ex-post) Nash equilibrium of the English auction.*

Proof. Corollary 5, Lemma 8, and Lemma 1 in Krishna (2001) imply that there is an ex-post Nash equilibrium of the auction. The strategy of bidder i in the sub-auction where there are A bidders active is $\beta_i = \sigma_i^{-1}$. By Lemma 10, the equilibrium is efficient. □

Remark 12. Like Krishna's, our results are compatible with situations where bidders do not drop out "in order." Krishna's Example 2, where a high-value bidder drops out before a low-value bidders is an example where $\mathbf{v}$ satisfies the OEP. Of course, the highest-value bidders are the last ones to drop out in equilibrium, as the OEP guarantees that the equilibrium is efficient.

4 A Weak Stolper-Samuelson Theorem

4.1 The Trade Model and the OEP

Consider an $n \times n$ trade-model: There are n production factors, n consumption goods, and constant returns to scale. Consumers supply their factor endowments inelastically—they do not consume production factors.

Let $\mathbf{x} = (x_1, \ldots x_n)$ denote a vector of factor prices; x_i is the price of factor i. Let $v_i(\mathbf{x})$ be the unit (average) cost of good i. Constant returns to scale implies that the cost of producing y_i units of good i is $v_i(\mathbf{x})y_i$. Let p_i denote the price of good i. There are zero profits in the production of good i if $p_i = v_i(\mathbf{x})$.

In the context of the trade model, we interpret the OEP as a relative-factor-intensity assumption: the OEP says that the production of good i is relatively more intense in the use of factor i. Consider first the case of 2 factors. The OEP says that, if the price of factor 1 increases and the price of factor 2 decreases, then the cost of good 1 must increase more than the cost of good 2 (or the cost of good 2 must decrease more than the cost of good 1). The OEP is an economic version of the technological assumption that the production of good 1 is relatively more intense in factor 1.

With more than 2 factors, all the OEP says is that *one* of the goods whose factor-price has increased must have a cost-increase that is larger than the cost-increase of any of the goods whose factor-price decreased.

4.2 The Result

An equilibrium in this model—where technology has constant returns to scale—is characterized by the zero-profit conditions. Say that a price-wage pair $(\mathbf{x}, \mathbf{p})$ is an *equilibrium* if $p_i = v_i(\mathbf{x})$ for all i. Assume that $\mathbf{v}$ is monotone increasing and satisfies the OEP.

Theorem 13. *Let* $(\mathbf{x}, \mathbf{p})$ *and* $(\mathbf{x}', \mathbf{p}')$ *be equilibria. If* $p_i < p'_i$ *for some good* i*, and* $p_h \geq p'_h$ *for all* $h \neq i$*, then* $x_i < x'_i$*, and* $x'_h < x_h$ *for at least one* $h \neq i$*.*

Proof. The statement of Theorem 13 is the statement of Corollary 4, adapted to the context of the $n \times n$ trade model. □

Theorem 13 is a weak, but global, version of the Stolper-Samuelson Theorem (Stolper and Samuelson, 1941). To see this, suppose first that there are two goods and two factors. In this case, Theorem 13 states that if a country opens up to trade and experiences, as a consequence, an increase in p_1, and that p_2 either decreases or stays the same, then the price of factor 1 will increase and the price of factor 2 will decrease. Thus the owners of factor 1 will gain, and the owners of factor 2 will lose, from opening up to trade.

Suppose now that there are more than two goods and factors. In this case Theorem 13 states that, if p_1 increases, and p_h either decreases or stays the same, for all other goods h, then the owners of factor 1 will gain, and the owners of at least one of the other factors will lose. We say that Theorem 13 is a *weak* version of the Stolper-Samuelson Theorem because it does not say that $x'_h < x_h$ for all $h \neq i$.[2]

4.3 Comparison with Stolper and Samuelson's version.

Theorem 13 delivers the message of the Stolper-Samuelson Theorem in considerable generality. We shall enumerate the differences between Theorem 13 and Stolper and Samuelson's statement:

1. Stolper and Samuelson's relative factor-intensity condition is stronger than the OEP. We elaborate on this below.

2. Stolper and Samuelson's conclusion is local; the conclusion of Theorem 13 is global.

3. Stolper and Samuelson require the cost function $\mathbf{v}$ to be differentiable, and that the Implicit Function Theorem be applicable.

4. Stolper and Samuelson's statement of the theorem is only true when $n = 2$ (see, for example, Chipman (1969)).

[2] We follow Chipman (1969) in using the "weak" modifier for this statement.

Let $n = 2$.

Stolper and Samuelson's statement requires that $\mathbf{v}$ satisfies the following condition:

Definition 5. $\mathbf{v}$ *satisfies the relative factor-intensity condition if* $\mathbf{v}$ *is* C^1 *in the interior of* $I\!R^2_+$, *and*

$$\frac{\partial v_1(\mathbf{x})/\partial x_1}{\partial v_1(\mathbf{x})/\partial x_2} > \frac{\partial v_2(\mathbf{x})/\partial x_1}{\partial v_2(\mathbf{x})/\partial x_2},$$

for all $\mathbf{x}$ *in the interior of* $I\!R^2_+$.

But the OEP is weaker than the relative factor-intensity condition. Let $\mathbf{v} : A \to I\!R^2$, where A is compact.

Proposition 14. *If* $\mathbf{v}$ *satisfies the relative factor-intensity condition, then it satisfies the OEP in the interior of* $I\!R^2_+$..

Proof. Let $\Delta(\mathbf{x})$ be the determinant of the Jacobian matrix of $\mathbf{v}$ at $\mathbf{x}$. The relative factor-intensity condition implies that $\Delta > 0$. The implicit function theorem implies that there is a C^1 map $\mathbf{x}(\mathbf{p})$ such that $\mathbf{p} = \mathbf{v}(\mathbf{x}(\mathbf{p}))$ for all $\mathbf{p}$ in the range of $\mathbf{v}$ (by compactness of A).

$$\begin{aligned} x_1' - x_1 &= \Big\{ \textstyle\int_{p_1}^{p_1'} \frac{\partial v_1}{\partial x_2}(\mathbf{x}(s, p_2))\Delta(\mathbf{x}(s, p_2))^{-1} ds \\ &\quad - \textstyle\int_{p_2}^{p_2'} \frac{\partial v_1}{\partial x_1}(\mathbf{x}(p_1', s))\Delta(\mathbf{x}(p_1', s))^{-1} ds \Big\} \\ x_2' - x_2 &= \Big\{ \textstyle\int_{p_2}^{p_2'} \frac{\partial v_2}{\partial x_1}(\mathbf{x}(p_1, s))\Delta^{-1}(\mathbf{x}(p_1, s)) ds \\ &\quad - \textstyle\int_{p_1}^{p_1'} \frac{\partial v_2}{\partial x_2}(\mathbf{x}(s, p_2'))\Delta^{-1}(\mathbf{x}(s, p_2')) ds \Big\} \end{aligned}$$

Let $x_1' > x_1$, $x_2' \leq x_2$. Then $p_1' \leq p_1$ and $p_2' \geq p_2$ is impossible, as $\partial v_i / \partial x_j > 0$ for all i and j. □

4.4 Comparison with other versions.

There is a large literature on generalizations of the Stolper-Samuelson Theorem. We shall not discuss the literature here; see Ethier (1984) for a survey.

The closest result to Theorem 13 is an application of the weak axiom of cost minimization (Ethier (1984)); but this application barely retains the economic content of the Stolper-Samuelson Theorem because, in trade theory, predicting *who* will win (and thus favor) an opening to trade, is crucial. Contrary to Theorem 13, the application of the weak axiom does not say which factor-prices change as a result of specific changes in goods prices.[3] The application of the weak axiom only gives the standard "average correlation" result between goods and factor prices: on-average-higher goods prices yields on-average-lower factor prices. On the other hand, the application of the weak axiom does not require assumptions on $\mathbf{v}$.

When $n = 2$, Samuelson (1953) also proved the Factor-Price Equalization Theorem: he proved that, if $\mathbf{v}$ satisfies the relative factor-intensity condition, $\mathbf{v}(\mathbf{x})$ has a global inverse, so factor prices are uniquely determined by $\mathbf{p}$. In the context of trade, this implies that all countries that share the same technology must have the same factor prices. This is, arguably, an empirically less relevant proposition than the Stolper-Samuelson Theorem, or the statement of Theorem 13.

When $n > 2$, the relative factor-intensity condition is not sufficient for the existence of a global inverse. Gale and Nikaido (1965) proved that, if $\mathbf{v}$ is C^1, and the Jacobian of $\mathbf{v}$ is everywhere a P-matrix—all the principal minors of $\mathbf{v}$ are positive—then $\mathbf{v}$ is globally invertible. But even if the Jacobian is everywhere a P-matrix, the Stolper-Samuelson Theorem need *not* hold (Chipman, 1969). Theorem 13 shows that our generalization of the factor-intensity condition suffices to give the Stolper-Samuelson result with $n > 2$. We do not address the problem of the existence of a global inverse.

5 Monotonicity of factor demands

Consider a price-taking firm that chooses a vector of production factors, $\mathbf{z} = (z_1, \dots, z_n)$, to maximize profits, $qf(z_1, \dots z_n) - \sum_{i=1}^{n} p_i z_i$, where q is the price of the firm's output, f is the firm's production function, and p_i is the price of factor i. Suppose f is monotone increasing and C^1.

For a vector of prices $\mathbf{p}$, let $\mathbf{z}(\mathbf{p})$ be a vector of factor demands.

Suppose the prices of, say, two factors, increase *by the same amount.* In general, the firm might find it profitable to employ more of one of these factors because they are substitutes, and/or because one of them has become relatively cheaper.

[3] The comparison with Jones and Scheinkman's (1977) "every factor has some natural enemy" result is similar.

Suppose, however, that $\mathbf{v}$ satisfies the OEP. Theorem 2 implies then that the firm will use less of both factors.

Proposition 15. *Let* $\mathbf{p}$ *and* $\mathbf{p}'$ *be price vectors such that* $\mathbf{z}(\mathbf{p})$ *and* $\mathbf{z}(\mathbf{p}')$ *are both interior. If* $p'_j - p_j = p'_h - p_h > 0$*, for* $j, h \leq k$*, and* $p'_j = p_j$ *for* $j \geq k$*, then* $z_j(p') \leq z_j(p)$*, for all* $j \leq k$*.*

Proof. Define $\mathbf{v}$ by

$$v_i(-\mathbf{z}) = q\frac{\partial f(\mathbf{z})}{\partial z_i}.$$

Then profit maximization implies, in an interior solution, that

$$\mathbf{p} = v(-\mathbf{z}).$$

$\mathbf{z}(\mathbf{p})$ and $\mathbf{z}(\mathbf{p}')$ are both interior, so $\mathbf{p} = \mathbf{v}(-\mathbf{z}(\mathbf{p}))$ and $\mathbf{p}' = \mathbf{v}(-\mathbf{z}(\mathbf{p}'))$. So $\mathbf{v}(-\mathbf{z}(\mathbf{p}')) \not\leq \mathbf{v}(-\mathbf{z}(\mathbf{p}))$. By revealed preference, $\mathbf{p} < \mathbf{p}'$ implies that $\mathbf{z}(\mathbf{p}) \not\leq \mathbf{z}(\mathbf{p}')$, so $-\mathbf{z}(\mathbf{p}') \not\leq -\mathbf{z}(\mathbf{p})$

The result now follows from Theorem 2. □

The textbook revealed-preference approach to factor demands implies that, when prices increase, the demand for *some* factors must decrease. Proposition 15 says more; it says which factor-demands will decrease. Of course, Proposition 15 requires stronger assumptions than the revealed-preference approach.

Proposition 15 requires that solutions be interior, and that the vector of marginal productivities $\partial f(\mathbf{z})/\partial z$ satisfy the OEP. For instance, if f is Cobb-Douglas, the resulting $\mathbf{v}$ will satisfy the OEP. One can ensure that solutions are interior by imposing conditions about the behavior of f close to the boundaries of $\mathbb{R}^n_+$.

One can also use monotone comparative statics methods (Milgrom and Shannon, 1994) to prove that factor demands are monotone. But monotone comparative statics requires that the inputs be complementary—concretely, that f be supermodular.

We present an example of a production function f that is not supermodular, and that satisfies the hypothesis of Proposition 15. The example shows that Proposition 15 indeed provides new results on the monotonicity of factor demands.

Example Let $h : \mathbb{R}_+ \to \mathbb{R}$ be C^1 and monotone increasing, and let $\alpha > 0$. Let

$$\mathbf{v}(x_1, x_2) = \begin{pmatrix} h(x_1) - \alpha x_2 \\ h(x_2) - \alpha x_1 \end{pmatrix}.$$

Then $\mathbf{v}$ satisfies the OEP: the OEP only has bite if $\mathbf{x}' \not\leq \mathbf{x}$ and $\mathbf{x} \not\leq \mathbf{x}'$. Say that $x_1' > x_1$ and $x_2' \leq x_2$. Then $v_1(\mathbf{x}') - v_1(\mathbf{x}) > 0$ and $v_2(\mathbf{x}') - v_2(\mathbf{x}) < 0$. Similarly if $x_1' \geq x_1$ and $x_2' < x_2$. So the OEP is satisfied.

Let $A \subseteq \mathbb{R}_+^2$ be a bounded open interval. By Thomas's Theorem (Thomas (1934), see also Hurwicz and Uzawa (1971)),

$$\frac{\partial v_1(\mathbf{x})}{\partial x_2} = \frac{\partial v_2(\mathbf{x})}{\partial x_1}$$

implies that there is a C^1 function $f : A \to \mathbb{R}$ such that $\partial f(\mathbf{x})/\partial \mathbf{x} = \mathbf{v}(\mathbf{x})$. Note that f is not supermodular, as $\partial f(\mathbf{x})/\partial x_1 = h(x_1) - \alpha x_2$ is decreasing in x_2. Further, if $\lim_{x\to 0} h(x) = \infty$, then factor demands will be interior because the resulting f satisfies the Inada condition.

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