Existence of GE: Are the Cases of Non Existence a Cause of Serious Worry?

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Abstract

In this work, we attempt to characterize the main theoretical difficulties to prove the existence of competitive equilibrium in infinite dimensional models. We shall show cases in which it is not possible to prove the existence of equilibrium and some others in which, however the existence of equilibrium can be proved, the equilibrium prices seem not to have natural economic interpretation. Nevertheless in pure exchange economies, most of these difficulties may be avoided by mild restrictions on the model. In productive economies new specifics problem appear, for instance non convexity of the production sets or non boundedness of the feasible allocation sets. To prove the existence and the efficiency of the equilibrium in productive economies we need some strong hypothesis about the technological possibilities of each firm.

Abstract

En este trabajo pretendemos caracterizar algunas de las más importantes dificultades para probar la existencia del equilibrio competitivo en modelos con infinitos bienes, esto es en modelos con bienes contingentes a los estados de la naturaleza o al momento en el que serán consumidos. Veremos que en algunos casos no es posible probar la existencia del equilibrio a menos de introducir en el modelo restricciones adicionales, no necesariamente fuertemente limitadoras del mismo. Mostraremos también casos en que si bien es posible probar la existencia del equilibrio, no es posible tener una buena interpretación económica para el concepto teórico o bien que presenta limitaciones que no nos permiten obtener previsiones sobre el desarrollo futuro de la econmía. Como es bien sabido no existe una dinámica satisfactoria que represente el porvenir en la teoría económica, este aspecto será presentado en el presente trabajo con una óptica diferente. Finalmente se analizarán las dificultades que los modelos con producción presentan mostrando las dificultades técnicas que desafían a los investigadores del área.

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1 Introduction

Looking for commodities as physical goods which may differ on time or in the states of the world in which they become available, and allowing infinite variation in these contingents, the generalization of the classical model of General Equilibrium to models of economies with infinitely many commodities looks natural.

As the extension of the classical general equilibrium model to an infinite dimensional setting gives answers to relevant questions of the economic theory, news theoretical challenges appear. For example cases where it is still now theoretically impossible to prove the existence of an equilibrium or where the mathematical interpretation or economical meaning of this equilibrium is not concrete or natural. Recent researches done by different authors, tried to obtain new theoretical tools or more general conditions for existence of the Walrasian equilibrium. For instance the condition in preferences defined in [Mas-Colell (86)] and known as properness, plays an important role compensating for the absence of interior points in positive cones in some Banach spaces. We shall analyze this and other conditions that play analogous roles.

In contrast with the finite dimensional cases where there exist a canonical commodity space, if the vector spaces are infinite dimensional there is not a canonical linear space. Different economic applications require models involving different (non-isomorphic) infinite dimensional linear spaces.

For instance, the usual finance models describe the commodities as being a stochastic process, so this suggests the space of square integrable functions L_2 as the commodity space, while in growth theory the typical space is the space of essential supremum bounded functions L_∞, in which each $x(t) \in L_\infty$ may be interpreted as an inter-temporal allocation.

As [Araujo, A.; Monteiro, P. K. (89)] shows for finance models the requirement of infinite marginal utility for consumption at zero, makes that generically on initial endowments Walrasian equilibrium doesn't exist.

In contrast, in growth theory for every strictly positive endowment the existence of equilibrium can be proved. As we shall see in what follows, in this case the problem is the concrete interpretation of the equilibrium prices in both senses, mathematical or economical.

In models that allow for many different characteristics, we are led to consider the Borel signed measures on a compact metric space K as the commodity space, where K represents the commodity characteristics and a positive measure x on K represents a commodity bundle comprising quantity of some of this characteristics, see [Mas-Colell, A. (75)].

It is important to notice that in infinite dimensional models, the *excess demand function* is typically not defined [Araujo, A. (87)]. Araujo argues that the excess demand function can be smooth only if the commodity space is a Hilbert space. The Negishi approach avoid most of the difficulties related with the non existence of demand function by means of the *excess utility function.* This approach allows to introduce differential methods in infinite dimensional models, (see for instance [Accinelli, E.(96)]), and gives a deep intuition inside of the structure of the equilibrium set. Nevertheless this approach is very dependent on utility functions and therefore on the properties of the preferences. The Negishi approach also depends on the Pareto optimality of the equilibrium, and thus on the topological properties of the commodity spaces that guarantee the existence of Pareto optimal allocations. Some examples of well behaved economies which have not Pareto optimal allocations are given in [Araujo, A. (85)].

In our work we will consider models in which preferences may be representable by utility functions. In finite dimensional economies, every continuous preference is representable by an utility function. However on infinite dimensional spaces this result may not be useful, because in general we lack separability [1], besides continuity countable boundedness must be added [2]. See [Monteiro, P.K. (87)]. Nevertheless in [Richard, S.F.; Zame, W. R.] it is proved that in a positive cone of a normed vector lattice, uniformly proper, continuous and convex preferences have a continuous utility function representation.

Prices will be elements of a dual space L^* of continuous linear forms on the topological vector space L, in which the commodity space is included. Mathematical possibilities and economic meaning of some properties of our models will depend only on the pair (L, L^*) selected.

2 Structural Characteristics of the Infinite Dimensional Models

As we said in the previous section different (non-isomorphic) infinite dimensional spaces arise in different economic situations. In contrast with the finite dimensional case we point out the following six characteristics of the infinite dimensional spaces, each of them arise a new theoretical challenge to give a positive answer to the question of the existence of equilibrium:

1. *Non-uniqueness of the topology.* While in finite dimensional vector spaces there is only one Hausdorff linear topology, [Aliprantis, C. D, Border, K.C.], an arbitrary infinite dimensional vector space admits more than one linear topology.

2. *The possibility of the existence of non-continuous linear functional.* In a finite dimensional space every linear functional is continuous, remember that the kernel of a linear functional is

[1] The non-separable L_∞ is a typical example.

[2] Let X be a set and $\succeq$ a preference relation on X. If $F \subset X$ and for all x in X there are y, z in F with $y \succeq x$ and $x \succeq z$ we say that F bounds X. If F can be taken countable, we say that $\succeq$ is *countably bounded*

a finite dimensional subspace and then it is closed, so continuity of linear functional follows. In infinite dimensional vector spaces it may exist non-continuous linear functionals. To observe this consider the vector space of every continuous function from $\Re$ in $\Re$ with compact support, $C_c(\Re)$. Let $x(t) = 0 \ \forall t \notin [a,b]$ be a continuous function in $[a,b]$, considering the norm given by the supremum in $[a,b]$. Consider the linear functional defined by the integral $\int_{\Re} : C_{[a,b]} \rightarrow \Re$. It is easy to see that this is a non continuous linear functional. To prove the claim consider: the function $x_n(t)$ that take value equal to 1 if $t \ \in [0,n]$, that is equal to 0 out of the interval $[-1, n+1]$, and is linear in $[n, n+1]$, in this case x_n is in $C_{[a,b]}$ but $\int_{\Re} x_n(t) = n+1$ thus, the linear functional is not bounded in the closed unit ball, and then it is not-continuous[3].

Recall that a positive linear functional should be interpreted as representing the economic concept of prices. In many commodity spaces positive linear functionals are automatically continuous, [4] nevertheless, not every Riesz space admits strictly positive linear functionals. This is the case of $\Re^N$, the vector space of all real sequences on N. This follows, because the topological dual of this space is the space of the sequences in $\Re^N$ which terms are zero, except for finitely many of them, see [Aliprantis, C. D, Border, K.C.] Remember that a functional p, is positive on a Riesz space E, if for each $x \in E^+$ (the positive cone of E), $< p, x > \geq \ 0$, and is strictly positive if $x > 0$ implies $< p, x >> 0$, for all $p \in (L^*)^+$.

Continuity of prices is in part a mathematical condition, and reflects the choice of topology, and in several settings is a weak requirement. Nevertheless the choice of topology has a strong economic meaning. It is possible the existence of equilibrium allocations supportable only by non-continuous linear functionals (prices) [5].

3. *Multiplicity of dual spaces.* A characteristic of the infinite dimensional economic models is that the pair commodity-price is described by a dual system $< L, L^* >$ where L is the commodity space and its dual L^* is the price space. We don't take care where L^* come from, we just need that the dual space itself be another vector space (so an infinite dimensional vector space may has several (non-isomorphic) dual spaces).

   A dual system is a pair $< L, L^* >$ of vector spaces together with a function $(x, x^*) \rightarrow < x, x^* >$, from $L \times L^*$ into $\Re$ satisfying:

[3] Moreover, every linear functional $x^* \in L^*$ attains it supremum on the unit ball of L if and only if L is a reflexive Banach space. See [Ciranescu, I.]

[4] This claim is truthful in a completely metrizable locally solid Riesz space.

[5] Nonetheless if we assume endowments strictly positives, and monotone preferences, support prices are $\tau-$continuous in the ideal generated by the total endowment w.

- The mapping $x^* \to < x, x^* >$ is linear for each $x \in L$.
- The mapping $x \to < x, x^* >$ is linear for each $x^* \in L^*$.
- If $< x, x^* >= 0$ for each $x^* \in L^*$, then $x = 0$.
- If $< x, x^* >= 0$ for each $x \in L$, then $x^* = 0$.

Each space of a dual pair can be interpreted as a set of linear functionals on the other.

A locally convex topology τ on L is said to be compatible with the dual pair $< L, L^* >$ if for each continuous linear functional f in (L, τ) there exists $x^* \in L^*$ (the topological dual of (L, τ)) such that $f(x) =< x, x^* >$.

4. *Lack on the continuity of the wealth map.* The wealth map $(x, p) \to < p, x >$ where $x \in L$ and $p \in L^*$, is jointly continuous in the finite dimensional case, in the infinite dimensional spaces it has sense to ask for the jointly continuity, and we will see that the answer depends on the topology of these spaces. Indeed this map is jointly continuous in the norm, but it is not jointly continuous if one of the spaces in the dual pair is given with its weak topology and the other one with its norm topology [6]. [Aliprantis, C. D, Border, K.C.]

5. *Attainable sets may not be compact.* The first problem is that some of the sets which are bounded in finite dimension may not be bounded in infinite dimensional setting. For instance if the commodity space is $L = L_\infty([0, 1])$, and the price $p \in L_1([0, 1])^+$ is not 0, then the budget set $B = \{x \in L_\infty([0, 1]) : < p, x > \leq < p, w >\}$ is never bounded, see [Mas-Colell, A.; Zame, W.R.].

   The second problem that arises is that the not norm-compactness of the unit ball is a characteristic of the infinite dimensional spaces (this is the claim of the Riesz theorem). Moreover, if the space L is not semi-reflexive [7], then there exists a bounded and closed set with the weak topology $\sigma(L, L^*)$ that it is not compact with this topology, see [Schaefer, H.H].

6. *Not supportability of convex sets.* Two disjoint non empty convex subsets, can be separated by a not zero continuous linear functional, provided one of them has an interior point, see this condition is always guarantee in the finite dimensional case, but it is no longer valid for infinite dimensional spaces.

[6] In particular if $x_n \to x, \sigma(X, X')$ in a reflexive Banach space X and $x'_n \to x', \sigma(X'', X')$, implies $< x_n, x'_n > \to < x, x' >$ then, X is finite dimensional.

[7] A locally convex space L is said to be semi-reflexive if $L = (L^*)^*$. We note that this property depends only on the duality (L, L^*), and hence is shared by all or by none of the locally convex topologies on L that are consistent with (L, L^*). (Semi-reflexivity and reflexivity agree for normed spaces)

Then taking C to be the set of consumption bundles strictly preferred to x with usual convex preferences, in infinite dimensional models, the existence of supporting prices is not guaranteed.

3 Examples of pure exchange economies with non existence of equilibrium

From now until section 6), in which we will introduce production, we restrict our attention to pure exchange economies. There are N consumers characterized by their consumption spaces in the positive cone of a locally convex, topological vector space. Each commodity space is endowed with an order structure, given by consumers preference relation $\succeq_i$. Preferences are complete pre-order, monotone and convex binary relations on consumption set. Each consumer has an initial allocation (*endowment*) w_i that belongs to the positive cone in his consumption space. Let us begin this section recalling the definition of the Walrasian equilibrium:

Definition 1 *A Walrasian or competitive equilibrium is a pair* $(p, x), x \in L, \ \ p \in L^*$ *such that* $x \in \mathcal{B}_i(p)$ *and* $\bar{x} \succ x$ *implies* $< p, \bar{x} > \geq < p, w_i >$, *where* $\mathcal{B}_i(p) = \{x \in L_+ : < p, x > \leq < p, w_i >\}$, *is the budget set of agent* i.

With the success obtained by the Black and Scholes formula the finance models have received a great stream of interest. Theorems on existence of equilibria for models like this were obtained by [Araujo, A.; Monteiro, P. K. (89)]. However in [Araujo, A.; Monteiro, P. K. (88)] it is showed that for these kind of models the set of the endowments for which the economy does not have an equilibrium is residual. This means that generically the most useful models in finance do not have a Walrasian equilibrium.

Araujo and Monteiro have proved that for economies with separable utilities $u_i : L_p^+ \rightarrow \Re, 1 \leq p < \infty$ (for (S, μ) a measurable space),

$$u_i(x) = \int v_i(x(s), s) d\mu(s), \tag{1}$$

where v_i is concave, monotone and differentiable, and such that the derivative at $(0, s), v'(0, s) = \infty$ for each s, the set of endowments that allows us to prove the existence of equilibrium is of first category [8] on L_p^+. This result, was generalized in [Monteiro, P.K.(94)], where the separability of the utility functions is turned out.

Nevertheless, if the endowments are positive ($w_i \in L_p^+ - \{0\}$) the condition that establishes that $v'(w(s), s)$ belongs to L_q $((1/p) + (1/q)$ is sufficient to prove the existence of equilibrium in

[8] A subset A of a vector space, is of first category, or meager if it is a countable union of nowhere dense sets.

the space generated by $[-w, w]$, $w = \sum_{i=1}^{n} w_i$ [9] (in finance setting $p = 2$.) (This condition is equivalent to the assertion that utilities are proper [10] in all individually rational weak optimum, see [Mas-Colell, A.; Zame, W.R.]). This result shows also, that even the supportability of the Pareto optimal allocation is not a typical property.

3.1 The possibility of emptyness of the Pareto optimal set

As we said before, some methods to study the existence of equilibria are strongly related with the existence of the Pareto optimal allocation, one of these is the Negishi approach.

Let us to begin this section with the definition of ordered vector space:

In a **Riesz space** E, *(which is a partially ordered vector space that is at the same time a lattice)*, an ordered interval is any set of the form:

$$[x, y] = \{z \in E : x \leq z \leq y\}. \tag{2}$$

If the dual pair $< E, E^* >$ is symmetric [11], where E^* is the normed dual space of E, then the intervals of E are $\sigma(E, E^*)$ compact. (*The norm dual L^* of a normed space $(L, \|\cdot\|)$is the vector space of all norm continuous linear functional on L equipped with the operator norm, also denoted $\|\cdot\|$. Recall that the norm dual of a normed space is a Banach space).*

Working with the dual pair of the bounded real sequences as the commodity space, and the space of absolutely summable sequences as the dual space (l_∞, l_1), [Araujo, A. (85)] proves that if we relax the assumption of continuity of the preferences with respect to the Mackey topology it's possible to obtain economies without Pareto optimal allocations. This results follows from the fact that the second dual space of the space of bounded sequences $(l_\infty, \tau_{Ma})''$, with the Mackey topology is isomorphic to the dual space of absolutely summable sequences, $(l_1)' = l_\infty$. Then with this topology, l_∞ is a semi-reflexive space. If preferences are weakly continuous, and the feasible set is bounded and closed for the weak topology, existence of the Pareto optimal allocations is equivalent with the semi-reflexivity of the commodity space [12].

There is an economically interesting property shaded by the Mackey topology: the property of *impatient* or myopic behaviour. A preference relation display an impatient behaviour if present consumption is preferred to the future consumption, and taste for future consumption diminishes as the time of consumption recede into the future.

[9]If in addition w is in the quasi interior of L_p^+ then equilibrium price extends to a continuous price on all of L and is an equilibrium price for the original economy

[10]This concept will be definite in section 5.

[11]A pair $< E, E^* >$ is a symmetric Riesz pair if and only if $< E^*, E >$ is a Riesz pair.

[12]Recall the following alternative characterization of semi-reflexive space: (L, τ) is semi-reflexive if and only every bounded subset of L is contained in a $\sigma(L, L^*)$ compact set.

Notice that the assumption of weak continuous utilities is a restrictive condition. By weakening a topology on a given space, its continuous functions set generally diminishes. Symmetrically: the stronger (finer) the topology on a given space L, the more continuous function there are [Narici, L.; Beckenstein, E].

It's important to describe now some natural topologies for infinite dimensional spaces, the most interesting are the weak topology $\sigma(L, L^*)$ and the Mackey topology $\tau(L, L^*)$. The weak topology is the weakest topology for which the map $x \rightarrow < x, x^* >$ is continuous, for each $x^* \in L^*$. In terms of convergence of nets, x_α converge to x in this topology if $< x_\alpha, x^* > \rightarrow < x, x^* >$, for each $x^* \in L^*$, for this particularity this topology is called the topology of pointwise convergence.The Mackey topology is the topology for which convergence $x_\alpha \rightarrow x$ means $< x_\alpha, x^* > \rightarrow < x, x^* >$, uniformly for $x^* \in \sigma(L^*, L)-$ compact subset of L^*. That is the net $x_\alpha \rightarrow x$ if for each $\sigma(L^*, L)-$ compact convex subset A of L^* we have: $sup\{| < x_\alpha - x, x^* > |, x^* \in A\} \rightarrow 0$. This topology is called topology of the uniform convergence.

If the topology on L is weak enough, then L^* can be very small, too small to be sensitive. One of the major results on duality theory, the Mackey-Arens theorem, establishes that:

All locally convex topology τ with the same continuous linear functional L^* lies between the weak topology and the Mackey topology. In other words, the dual of $\sigma(L, \sigma(L, L^*))^*$ is just L^*, and the dual of $\tau(L, \tau(L, L^*))^*$ is L^* too, even though $\tau((L, L^*)$ is generally a finer topology than $\sigma(L, L^*)$. Moreover $\tau(L, L^*)$ is the finest topology for L which leaves L^* as the dual space of L. Clearly the finest topology is the richest in continuous functions, [Narici, L.; Beckenstein, E]. It follows from the Hahn- Banach theorem that all equivalent topologies have the same closed convex sets, and the same weakly bounded sets too. A set $A \in L$ is weakly bounded if for each $x^* \in L^*$ the set $\{< x_\alpha, x^* > x \in A\}$ is bounded in $\Re$.

As a corollary of the above claim it follows that all topology consistent with a given dual pair has associate the same set of upper semi-continuous quasi-concave functions. The proof is a straightforward conclusion of the fact that u is a quasi concave function if and only if, the set $\{x : u(x) \leq \alpha\}$ is convex for each α. If these sets are closed in some topology there are closed in all consistent topology.[13]

So in the above cited work, Araujo proves that the *continuity with respect to the Mackey topology is the best assumption of this kind, that guarantees the existence of a Pareto optimal*

[13]Nevertheless the weak topology is really different from a strong one. To see this consider the sequence $\{e_i\}_{i=1}^{\infty}$, in the l_2 space, where e_i is defined by $e_{ij} = 0$ if $i \neq j$ and equal to one in otherwise, $j = \{1, 2...\}$. From the Riesz Representation theorem it follows that for all linear functional on l_2, there exist an element $a \in l_2$ such that $f(e_i) \;\; =< e_i, a > = a_i$, then $f(e_i) \rightarrow 0$ and so, the weak convergence follows. However this sequence doesn't converge in the norm topology.

allocation.

To show an example of "well behaved" economy without Pareto optimal allocation, let us consider the *possibility utility set:*

$$\mathcal{U} = \{(u_1(x_1), \ldots, u_n(x_n)) \in \Re^n : (x_1, \ldots, x_n) \text{ is a feasible allocation}\}, \tag{3}$$

an n-tuple $(x_1, \ldots, x_n)$ is called a feasible allocation whenever $x_i \geq 0$ holds for each i and $\sum_{i=1}^n x_i \leq w$ where w is the total endowment.

Note that if each consumer has monotone preferences, then the set U is bounded above by $(u_1(w), \ldots, u_n(w))$. The weak compactness of the interval $[0, w]$ together with quasi-concavity and Mackey upper semi-continuity of each u_i implies that the economy satisfies the closedness condition [14]. But the converse is not true. To see this, consider the following example:

Example 1 *Consider the exchange economy with Riesz dual system* $(C[0,1], ca[0,1])$*, with two consumers with utility functions* $u_1(x) = \int_0^1 x(t)dt$ *and* $u_2(x) = \int_0^1 \sqrt{x(t)}dt$*, and total endowment* $w = \mathbf{1}$*. (Keep in mind that* $ca[0,1]$ *is the norm dual of* $C[0,1]$ *equipped with de sup norm.)*

The interval $[0, \mathbf{1}]$ is not weakly compact. Nevertheless the utility space of this economy is the set: $\mathcal{U} = \left\{(u_1, u_2) \in \Re^2_+,\ u_1 + (u_2)^2 \leq 1\right\}$, which is a closed set.

As we said above, the weak compactness of the order interval $[0, w]$, is a sufficient condition for the existence of a Pareto optimal allocation, but in [Mas-Colell, A. (75)] a weaker condition was given:

For each exchange economy that satisfies closedness condition, the set of Pareto optimal allocations is non-empty. This claim follows as a consequence of the Zorn lemma. [15]

The following example shows that without upper semi-continuity in Mackey topology the utility possibility set may be not closed and then a Pareto optimum may not exists, [Araujo, A. (85)].

Example 2 *Consider an exchange economy, with dual pair* (l_∞, l_1) *and utility functions:*

$$u_1(x) = \sum_{n=1} \frac{x_n}{2^n} \quad \text{and} \quad u_2(x) = \lim_{n \to \infty} inf x_n$$

and endowments $w_1 = \mathbf{1};\ w_2 = \mathbf{1}$.

[14]This claim follows from the fact that Mackey upper semi-continuity of a quasi-concave function imply weak upper semi-continuity, then $limsup_\alpha u_i(x_\alpha) \leq u_i(x)$ for each net x_α weakly convergent to x. Then for a feasible x if $u_i(x_\alpha)$ converge to η_i, $\eta = (\eta_1, \ldots, \eta_n) \in$ U.

[15]For each allocation x such that $u(x)$ belong to U consider $\mathcal{C}_x$ the set of all comparable allocations with x. Let us now to consider the nondecreasing sequence $u(x_\alpha) \geq u(x)$ in $\mathcal{C}_x$. As the utility possibility set is a bounded real set, then closedness implies compactness of this set then, there exist $z \in \mathcal{U}$, such that $u(x_\alpha) \uparrow z$, consider now the feasible allocation y such that $u_i(y_i) = z_i$. This is an upper bound for the order given by preferences in the sequence x_α. Then by the Zorn lemma, there exist a maximal element in $\mathcal{C}_x$, this is a Pareto optimal allocation.

It is easy to see that both utility functions, u_1 and u_2 are concave, monotone and norm continuous functions. In order to establish that u_2 is not Mackey upper semi-continuous, let us consider $x_n = (0, 0, ...0, 1, 1, ...)$, where there are zeros in the first n positions, note that $x_n \to 0$ in the Mackey topology, while $\lim_{n\to\infty} u_2(x_n) > u_2(0)$.

The utility possibility is:

$$U = \{(a_1, a_2) \in \Re^2 : a_1 < 2,\ a_2 \leq 2,\ \text{ or } a_1 = 2,\ a_2 = 0\}$$

is not closed.

Then assuming continuity of preferences with respect to a stronger topology that the Mackey topology, it is possible to obtain a large class of economies without Pareto optimal allocations and then without equilibrium.

In [Aliprantis, C.D.; Brown, D.J.; Burkinshow, O.] it is proved that if the consumers exhibit impatient behavior [16] then the closedness condition is satisfied. So the impatient behavior [17] is enough to guarantee the existence of a Pareto optimal allocation. In [Brown, D., Lewis, L] it is proved that the Mackey continuity of preferences implies impatience on the part of the consumers. The result of Araujo above cited, and the this later, show that if the dual system considered is (l_1, l_∞), Mackey topology is the strongest topology for which all upper semi-continuous preference is impatient.

3.2 Topology and equilibrium prices existence

The following example shows that the existence of the equilibria of the economy may be related with the dual pair considered That is, on an economic model with a fixed dual a pair it is possible to prove the existence of an equilibrium price such that this price has no sense if we consider the same model but with another dual pair.

Example 3 *The exchange economy with utility functions defined by:*

$$u_1(x) = \int_0^1 tx(t)dt;\ \ u_2(x) = \int_0^1 (1-t)x(t)dt$$

and endowments given by $w_1 = w_2 = \frac{1}{2}\chi_{[0,1]}$, *has not Walrasian equilibrium if the dual pair is* $(L_p[0,1], C^1[0,1])$ *and has equilibrium when we consider the dual pair* $(L_p[0,1], C[0,1])$.

[16] a consumer display an impatient behavior if for any x, y, and z if x is preferred to y then x is preferred to $y + \bar{z}$ where $\bar{z}$ is defined by $z_{nt}^{-} = 0, 1 \leq t \leq n$ and $z_{nt}^{-} = z_{nt}$, for $t < n$.

[17] In terms of the Growth Theory, impatience is equivalent with the fact that consumers discount the future.

In fact $p(t) = max\{t, 1-t\}$ is the only one linear functional on $L_p[0,1]$, such that $x_i \succeq x$ implies $< p, x_i > \geq < p, w >, i = 1,2$ where $x_1 = \chi_{[\frac{1}{2},1]}$ and $x_2 = \chi_{[0,\frac{1}{2}]}$.

Notice that as $p \notin C^1[0,1]$, then the allocation $(\chi_{[\frac{1}{2},1]}, \chi_{[0,\frac{1}{2}]})$, is not a Walrasian equilibrium with respect to the dual pair $(L_p[0,1], C^1[0,1])$.

But $p \in C[0,1]$, then the allocation $(\chi_{[\frac{1}{2},1]}, \chi_{[0,\frac{1}{2}]})$, is a Walrasian equilibrium with respect to the dual pair $(L_p[0,1], C[0,1])$.

4 It is enough to prove the existence of an equilibrium?

In this section we show some cases where the existence of equilibrium is guaranteed, but the economic interpretation or possible applications of this concept may be not clear, or not totally satisfactory.

Existence of not priced commodities. As it is familiar for finite dimensional models, a price, or a price system may be considered as a positive, continuous and linear functional defined for each commodity. It is well know that, for infinite dimensional spaces continuity of a linear functional depends on the topology on the space. For the economic theory, the restriction that implies to consider only economic models where the consumption sets are subsets of topological spaces in which all positive linear functionals are continuous, is a mild restriction. However to require defined prices for each commodity conceivable, may be a very restrictive condition, because not every commodity is present in the market at any time.

To clarify this topic consider the following example:

Example 4 *Suppose that the economy is defined in a topological vector lattice L, on a measure space (S, μ) with topology τ and that the consumption set is $X_i \subset L$ for each individual i and that the utility possibilities set U is closed.*

Let us define the set $L(w) = \{x \in L : |x| \leq \lambda w, \text{for some } \lambda > 0\}$, where w is the total endowment. Note that $L(w)$ contains all feasible consumption bundles. If we consider the restriction of the economy to $L(w)$, it is possible to obtain an allocation and a price of equilibrium, restricted to this set see [Mas-Colell, A.; Zame, W.R.].

The search of equilibria in $L(w)$ is much easier than in L because in $L(w)$ with the norm given by:

$$\|x\|_\infty = inf\{\lambda > 0 : |x| \leq \lambda w\}^{18},$$

the positive cone has not empty interior.

[18] By $|x|$ we denote $sup(x, 0) + sup((-x), 0)$.

If $L = L_\infty(S,\mu)$ and w is bounded away from 0, then $L = L(w)$. In general $L(w)$ is much smaller than L : for instance, if $L = L_p([0,1],\mu])$ with $1 < p < \infty$, $L(w) \subset L_p([0,1],\mu])$. If $w = \mathbf{1}$ then, $L(w)$ is precisely $L_\infty(S,\mu)$, and as $L_\infty(S,\mu) \subset L_1(S,\mu)$, then $p \in L_\infty(S,\mu)^*$. So, we assign finite prices only to every commodity bundle in $L_\infty(S,\mu)$ (commodities in the market) but we do not assign finite prices to all element of the consumption space, i.e., some conceivable commodities are left unpriced.

It is natural now to ask about the continuity of p in τ and its extensibility to L :

In [Yannelis, N.C.; Zame, W.R.)] it is proved that if preferences are F-proper [19] then, p is continuous, and if $L(w)$ is dense in L (i.e. if and only if w is a quasi-interior point of L^*_+) then the price p has an unique continuous extension to all L [20].

No clear interpretation for equilibrium prices concept. The following example shows a case in which there is not a natural interpretation for equilibrium prices in both economic or mathematical sense.

Example 5 *Suppose an economy with consumption set contained in l_∞ and its preferences are continuous in a topology τ stronger than the Mackey topology $\tau(l_\infty, l_1)$. In [Araujo, A. (85)] it is proved that there exist a linear $\tau-$continuous linear functional $p \in l^*_\infty$ that is not in l_1.*

Consider one consumer economy, with $w = (1,1,\ldots,1)$ and utility function $u : l_\infty \to \Re$ defined by $u(x) = \lim inf\ x(t)$. This is a concave, monotone and norm continuous function, but it is not Mackey continuous. So, there is a price $p \in l^*_\infty$, such that $< p, x > \geq < p, w >$ whenever x is at least so good as w, i.e. $u(x) \geq u(w) = 1$. This p can not belong to l_1. To see this, consider $x_k \in l_\infty$ such that $x_k(t) = 0$ if $t < k$, and $x_k(t) = 2$ if $t \geq k$. Then $u(x_k(t)) = 2 > u(w) = 1$, but if $p \in l_1$ the $< p, x_k > \to 0$ while $< p, w >>\ 0$.

Then, how do we describe these equilibrium prices? Let us denote by $ba(2^N)$ the space of all signed charges of bounded variation on the σ-algebra 2^N of all subsets of N, the natural numbers, by $ca(2^N)$ the σ-additive signed measures of $ba(2^N)$, and by $pa(2^N)$ the *purely finite additive* signed measure, then $l^*_\infty = ba(2^N) = ca(2^N) + pa(2^N)$, The spaces l_1 and $ca(2^N)$ are isomorphically equivalents and analogously l_1^d the complementary set of the l_1 space in l^*_∞, and $pa(2^N)$.

It can be proved that the elements of the set of purely finite additive measures are limit points of sequences e_n that assign mass one in $\{n\}$, thus there are measures zero-one, that is for each $A \subset N$, $\mu(A) = 0$ or $\mu(A) = 1$, see [Aliprantis, C. D, Border, K.C.]. Since utility depends only

[19] This concept will be defined later in section 5.

[20] In a Riesz space (E, E^*), $x \in E_+$, is a quasi interior point if $< x, x' >>\ 0$ for each $0 < x' \in E^*$. A quasi interior point is also called strictly positive, written $x >> 0$.

on what happens at infinity, the fact that the equilibrium price is a measure that has all its mass concentrated in the infinite is not surprising, but its existence depends on the Zorn lemma, (a non-constructive proposition). Also, it can be proved that a purely finite additive measure vanishes in finite sets. Clearly it has not a concrete economic intuition.

Araujo proved that to obtain prices in l_1, we need Mackey continuity of preferences in the dual system (l_∞, l_1). For stronger topologies, equilibrium prices could be in l_1^d the complementary set of l_1, which is isomorphically equivalent to the set of purely finite additive measures.

On the other hand, when the consumption set is included in L_∞ the Mackey continuity of preferences it is not enough to yield prices in $L_1(S, A, \mu)$. We must require, Mackey continuity, strictly monotone preferences, and that consumption sets coincide with the positive orthant $L_\infty(S, A, \mu)^+$, see [Bewley, T.]. Prices in $L_\infty(S, A, \mu)^* = \mathcal{M}(S, A, \mu)$ (the space of finite countably additive measures on the set S), that not belong to $L_1(S, A, \mu)$ have not natural economic interpretation. ($ba(S, A, \mu)$ *is the bounded finitely additive measures in A which vanish on sets of μ measure* 0).

Impossibility of predictions. An important question arises when we attempt to predict future states of an economy: is there local uniqueness of equilibrium? If the equilibria set of an economy has exactly one element, we would have a complete explanation of the state of the economy in the Walrasian framework. However global uniqueness requires very strong assumptions, generally this exigency is replaced by one of local uniqueness. The local uniqueness property guarantee the existence of a discrete set of equilibria, otherwise the slightest error of observation on the data of economy might lead to an entirely different set of predicted equilibria. Local uniqueness guarantee that in a neighborhood sufficiently small of the equilibrium price there is not another equilibrium price. Using differential topology, G. Debreu has given a satisfactory answer to this question for finite dimensional models, [Debreu, G. (74)]. An extension to a infinite dimensional models is given in [Accinelli, E.(96)].

For inter-temporal economies with separable utility functions, using the Negishi approach in [Accinelli, E., Puchet, M.] it is shown that there is not local uniqueness of the equilibrium path. The equilibrium set doesn't depend nicely on parameters and the possibility of predictions and of comparative statics analysis, both are lost. This impossibility to forecast or characterize the future state of the economy does not depend on the precision with which we can observe the parameters (endowments): it is a typical characteristic of the model. To obtain this negative result it is sufficient the existence of a singular endowment, that is a $w = (w_1, w_2, \dots w_n)$ for which the Jacobian of the excess utility function is a singular matrix. To avoid singularities we need strong assumptions on the utility functions.

As it is well know, the convexity of preferences appears as the only serious assumption needed to obtain existence of equilibrium, nevertheless this is not sufficient to obtain good interpretations or predictions in economic models.

5 Equilibria and Quasi-equilibria

In many cases we must add some additional conditions in the model to guarantee the existence of equilibria, for instance endowments strictly positive, or continuity of the utility functions. Quasi-equilibrium is a weaker concept than the one of equilibrium but, in some models where the existence of equilibria is not guarantee, it is possible to prove the existence of a quasi-equilibrium

If market prices are equilibrium prices they may be consider as a measure of the scarcity and, knowing equilibrium prices, each agent interacts with the market rather than with each other. To be a good signal, a system of prices must be, at least, clear about the possibilities that each agent has to obtain commodities in the market . These possibilities are restricted by his budget set.

To be an approximately good substitute for the equilibrium concept, quasi-equilibrium would be a good signal in the sense of the above statement. Then, at least, this concept would allows each agent to know the commodity bundles that he will be able to obtain in the market and which of them are out of his budget possibilities.

Let us now introduce the concept of quasi-equilibrium:

Definition 2 *Let be an exchange economy in which $x = (x_1, x_2, ..., x_n)$ is an allocation and p is the price system, the pair (p, x) is a quasi-equilibrium if $\bar{x} \succeq_i x_i$ implies $< p, \bar{x} > \ \geq \ < p, w_i >$. for all agent i. For an equilibrium $< p, \bar{x} > \ > \ < p, w_i >$, so $\bar{x}$ is unreachable.*

Two basic properties for a price p supporting a quasi-equilibrium allocation x are:

i) $< px_i >=< pw_i >$ for each i and

ii) if one preference is monotone, then $p \geq 0$.

So, by the definition, either a quasi-equilibrium or an equilibrium price, is a support for the allocation x, in the sense that if $\bar{x} \succeq_i x_i$ implies $< p\bar{x} >\geq< px_i >$.

The existence of an extremely desiderable [21] bundle for each consumer implies that a Walrasian equilibrium is necessarily a quasi-equilibrium, but the converse is not true: see for a counterexample section 1.6 in [Aliprantis, C.D.; Brown, D.J.; Burkinshow, O.]. Nevertheless for an exchange economy with strictly positive endowments w and continuous preferences, the quasi-equilibrium

[21] Recall that a vector v is said to be an extremely desirable bundle if $x + \alpha v \succ x$.

allocation is a maximal element in the budget set $B_i(p) = \{x \in L :< px > \leq < pw_i >\}$. If the quasi-equilibrium price is positive and $< pw >> 0$ then the quasi-equilibrium allocation is weakly Pareto optimal.

On finite dimensional models, the existence of a support price for a rational Pareto optimal allocation is a straightforward application of the convex separation theorems. Unfortunately, infinite dimensional spaces do not possess this property because, as we said in section 2, convex sets in infinite dimensional spaces may have empty interior (this is the case of the positive cone in L_p; $1 \leq p < \infty$). The property known as *properness* allow us to work in absence of interior points in the positive cone, this concept appeared first as cone condition in [Chichilnisky, G., Kalman, P. J.]. The following definition is given in [Mas-Colell (86)]:

Definition 3 *Let E be a Riesz space on which τ is a linear topology. We say that the preference relation $\succeq$, defined on the consumption set $X \subset E$ is proper at x with respect to the vector v, if there is an open cone Γ_x at 0 containing v, such that $x - \Gamma_x$ does not intersect the preferred set ($\{x' \in X : x' \succeq x\}$) i.e if $x' \succeq x$ then $x - x' \notin \Gamma_x$.*

This property may be interpreted considering a bundle set v as desirable, in the sense that the loss of an amount $\alpha v, \alpha > 0$, can not be compensated by an additional amount αz for any commodity z if z is not sufficiently big.

When preferences are convex, properness of $\succeq$ at x with respect to v is equivalent to the existence of a price $p \in E^*$ which supports the preferred set (*the better than x set*) and verify that $< p, v >> 0$.

A related notion was introduce in [Yannelis, N.C.; Zame, W.R.)]. We say that $\succeq$, is F-proper (F- for forward) at $x \in X$ if there is an open cone Γ_x at 0 containing v, such that $x + \Gamma_x \cap X \subset \{x' \in X : x' \succeq x\}$, i.e if $z \in \Gamma_x$ and $x + z \in X$ then $x + z \succeq x$.. In general properness and F-properness are incompatible conditions, nevertheless both condition are easy to check and hence have potential applications.

In [Araujo, A.; Monteiro, P. K. (88)] it is proved that for economies with separable utilities, and in which $L = L_p$ is the commodities space, properness is equivalent to the existence in the dual space L_q of the right hand derivative of $v(\cdot, s)$, (see equation (1)).

The following theorem is proved in [Mas-Colell, A. (75)]. *If in a pure exchange economy preferences are uniformly τ proper* [22] *and the order interval $[0, w]$ is weakly compact, then the economy has quasi-equilibrium.*

[22] A preference is uniformly proper if we can choose the same properness cone in each $x \in L^+$.

Moreover, if the total endowment is strictly positive and utilities are continuous functions, with the above hypothesis we obtain that the economy has a Walrasian equilibrium.

Existence of quasi-equilibrium don't require continuity of the utilities, the closedness condition of the utility possibility set (provided with the supportability of every weak optimum) is sufficient to guarantee the existence of a quasi-equilibrium, see [Mas-Colell, A.; Zame, W.R.].

Remember that upper semi-continuity and quasi-concavity of utilities are implicit in the assumption that the utility possibility set is closed. Full continuity will be required to prove the existence of equilibria.

For economies with separable utilities and strictly positive endowments, the existence of quasi-equilibria follows with the weaker assumption that the properness property is satisfied only at initial endowments, or in some rational allocation, see [Araujo, A.; Monteiro, P. K. (89)]. The loss of working in such a way is to give up the original commodity space and to work only with the feasible allocations set.

An easy prove of the existence of the Walrasian equilibrium for economies with separable utility functions, using the K.K.M theorem and the excess utility function is given in [Accinelli, E.(94)]

According to our above statement, in exchange economies, convexity of preferences is the more serious hypothesis to prove the existence of the equilibrium or quasi-equilibrium. Continuity of the utility function, strictly positive endowments and the separability of convex sets, may be in some cases avoided, but convexity of preferences is an unavoidable condition to be sure of the equilibria existence.

6 Productive Economies

The equilibrium analysis is technically more demanding for productive economies than for pure exchanges economies, even in finite dimensional cases. Taking care of the supportability and compactness issues, the existence of general equilibrium is guaranteed in pure exchange economies, but new specific difficulties appear in productive economies.

Even in finite dimensional models, to guarantee the existence of equilibria we must do some restrictive considerations in the technological possibilities. Also in cases in which it is possible to be sure of its existence, the question of its efficiency appears as a problem with not trivial solution (in some cases it is possible to obtain an equilibrium allocation that is not Pareto superior).

The critical technological assumption regarding firms is that their production sets are convex. As it is well know this expresses the notion of constant or diminishing returns to scale. Convexity of the production set can be derived by the primitives concepts *additivity and divisibility.* In models in which these hypothesis hold and under the classical hypothesis about the behavior of each agent and his consumption set, for finite dimensional models, the existence of equilibria follows as a corollary of a fixed point theorem, and its Pareto optimality may be guaranteed. While the additivity assumption seems hard to reject, divisibility assumption is much more debatable, both theoretically and empirically. Hence the main source of non convexity appears related to a failure in this assumption. Non convexity in much cases is consequence of increasing returns to scale, see [Mas-Colell (87)].

As it is well known firms with increasing returns to scale may behave as monopolies, and then they could be settle prices, affecting the prices and the optimality of the possible equilibria.

In presence of non-convex technologies the identification between equilibrium and optimum will no longer hold. Thus the existence of equilibrium and the analysis of its optimality become very different questions.

When production sets are non-convex, prices can be understood as a regulation policy aiming Pareto efficiency.There is no way of allocating efficiently the resources through a price mechanism in the presence of increasing returns to scale: this aiming requires taking decisions with distributive impacts, then some consumer may feel that he is paying too much for the optimality.

Moreover the following discouraging result holds:

Each economy has a non-empty core, if and only if the aggregate production set is a convex cone, see [Quinzii, M].

The idea behind the core is the social stability. If there is an allocation in the core of an economy a group of agents that can do better on their own does not exist. When the core is empty, the possibility of the intervention of some authority seem to be natural, but it is not the subject of this work.

Now we will study productive economies in the setting of infinite dimensional models. Formally we have the following definition of a productive economy:

Definition 4 *A private ownership productive economy is a set:*

$$\mathcal{E} = \{X_i, w_i, u_i, Y_j, \theta_{ij}, i = 1, 2, ..., n; j = 1, 2, ...m\}$$

where:

- i) $X_i \subset L$ is the consumption set, and L is a topological vector space. In L it is defined a topology τ consistent with the dual system (L, L^*).
- i) each consumer is characterized by his endowments w_i and by his utility function u_i,

- ii) there are m producers indexed by j each of whom has a production set $Y_j \subset L$.

- iii) The real number θ_{ij} represents the share of consumer i to the profit of producer j, $0 \leq \theta_{ij} \leq 1$ and $\sum_{i=1}^{n} \theta_{ij} = 1$, for all j. That is, the firms are owned by the consumers.

An allocation $(x, y) = (x_1, ..., x_n, y_1, ..., y_m) \in \Pi_i X_i \times \Pi_j Y_j$ is feasible if $\sum_{i=1}^{n} x_i \leq w + \sum_{j=1}^{m} y_j$, where w is the aggregated endowment.

In a private ownership productive economy the **wealth** of each consumer is $\gamma_i(p) =< pw_i >$ $+ \sum_{i=1}^{n} \theta_{ij} < pY_j >$,where p is the vector of prices.

To prove the existence of an equilibrium for this kind of models, we must be careful with the problems that arise in an infinite dimensional pure exchange economy plus the new ones that appear by the introduction of the production sets.

A quasi-equilibrium for E is a feasible allocation (x, y) and a linear functional $p : L \rightarrow \Re, p \neq 0$, such that:

- (a) $< p, x_i > \leq < pw_i > + \sum_{i=1}^{n} \theta_{ij} < pY_j >$, for all i.

- (b) $< p, y_j >= max\ pY_j$ for all j.

- (c) If $z \succeq_i x_i$ then $< p, z > \geq < pw_i > + \sum_{i=1}^{n} \theta_{ij} < pY_j >$, for $i \in (1, 2, ...n)$.

Moreover if $z \succ_i x_i$ implies $< p, z >>< pw_i > + \sum_{i=1}^{n} \theta_{ij} < pj_j >$, for $i \in (1, 2, ...n)$, also we say that (x, y, p) is an equilibrium.

To prove, in infinite dimensional models, the existence of equilibrium, the boundedness assumptions that are typically used in finite dimensional problems to obtain compactness of the feasible allocations are not enough, see section 1) item 5). Nevertheless, for a productive economy with a symmetric Riesz dual pair $< E, E^* >$, if all production sets Y_j are order bounded from above, then each feasible set is weakly compact.

In some cases, to prove the existence of an equilibrium, the compactness for the feasible sets is directly assumed, see for instance [Mas-Colell, A.; Zame, W.R.].

The supportability problem disappears if we suppose that the production set is a non empty positive cone, for instance l_∞^+. However, as in pure exchange economies, in productive economies the problem of the meaning of the equilibria prices appear. Mackey continuity and monotone preferences are not enough to prove the existence of equilibrium prices with an economical sense: we need to admit, in addition, closed and weak compactness of the production set in the $\sigma(l_\infty, l_1)$ topology.

For production sets in which the positive cone has empty interior, the failure of supportability may entail non existence of quasi equilibria. Once again, the concept of properness appears as a good substitute for the Hahn-Banach theorem.

In [Araujo, A.; Monteiro, P.K. (93)] it is shown that in many cases, including $L_p, 1 \leq p \leq \infty$, and in which measures are defined on a compact set, it is possible to prove the existence of an equilibrium with economic meaning, that is, an equilibrium price in L_1. To obtain these results the following hypothesis were stated:

- For each firm, the technological set Y_j is a convex Mackey closed subset of the consumption space.
- Y_j is a pointwise Mackey proper production set.
- The allocations set is bounded.
- In which concerns the consumer, preferences are norm continuous, consumption spaces are pointwise propers [23]and endowments are strictly positives.

Pointwise proper is a weaker condition than the uniform properness condition (considered in [Mas-Colell (86)]), but the original commodity space is given up and only the feasible set is considered, then it could exist a not priced commodity.

In the above cited work it is proved a general extended equilibrium for separable Banach lattices with order norm continuous E, see [Peresini, A.,L.]. To prove the equilibrium existence P.K. Monteiro proves that there exists a linear bijection $\theta : L_\infty \to E$.

We shall see some examples to clarify the above statements.

Example 6 *Suppose that* $\mathcal{E} = \{X_i, \succ_i, Y_j, \theta_{ij}, i = (1, 2, ..., n); j = (1, 2, ..., m)\}$ *is an economy with commodity space* $E = L_p, p < \infty$ *or* $E = \mathcal{M}(\Omega)$ *where* $\mathcal{M}(\Omega)$. *Suppose that:*

- *Preferences are convex and norm continuous; and* X_i *is a closed and convex set of* E_+. *Preferences on* X_i *are norm proper, that is for each* $x \in X_i$ *there exists* $v \in E$ *and* U_x *a neighborhood of zero such that* $x' \succeq x$ *for* $x' = x + tv - tz, z \in U$, *and* $t \geq 0$.
- *On the producers side* Y_j *is closed and convex,* $0 \in Y_j$, $Y_j - E_+ \subset Y_j$. Y_j *is a pointwise proper production set, i.e., for each* $y_j \in Y_j$ *there exist,* $v \in E_+$ *such that* $h = y - tv + tz$ *where* $t > 0$ *and* $z \in U$ *(a neighborhood of zero) is such that* $h^+ \leq y^+$ *then* $h \in Y_j$.

[23] A set X is pointwise proper if for all $x \in X$ and if (v_{ix}, U_{ix}) are properness constants then $x + v_{ix} \succ x$.

- *The set of feasible allocations restricted to $K(w) = \cup_{r>0}[-rw, rw]$, is bounded in $K(w)$. That is, there exists $b \in K(w)$ such that for all feasible allocation $(x_1, x_2, ..., x_n; y_1, y_2, ...y_m)$, $x_i \leq b, and y_j \leq b$.*

Then $\mathcal{E}$ restricted to $K(w)$ has an equilibrium which prices are in L_1.
See [Araujo, A.; Monteiro, P.K. (93)]

Contrary to intuition the worst case to prove the existence of an equilibrium is the L_∞ [24]. As we already have show, to prove the existence of equilibrium prices with economic meaning, we have to assume that preferences are Mackey continuous, and this assumption implies impatient behavior of the part of agents, which leaves many interesting preferences outside equilibrium theory. As the above example show in $L_p, 1 \leq p < \infty$ or M one does not need special assumptions on continuity of preferences, (as some kind of weak continuity), norm continuity is enough. To prove the existence of quasi-equilibrium in L_∞ see [Bewley, T.].

7 Conclusions

We wish to begin this last section with a Plato's remark on the duplicate cube problem, that seems particular apt for our dissertation: *"It must be supposed, not that the gods specially wished this problem solved, but that he would have the Greeks desist from war and wickedness and cultivate the muses, so that, their passions being assuaged by philosophy and mathematics, they might live in innocent and mutually helpful intercourse with one another".*

In General Equilibrium Theory there are two main questions, one of them is the problem of the existence of the equilibrium and the other one is in the cases when the equilibrium exists, about its properties and interpretations, in the first place the questions related with its efficiency and in second place the question about its predictive possibilities. Again, the question is:

Is the knowledge of the existence of equilibrium enough to know the behavior of an economy ?

As it is well known General Equilibrium Theory has not a dynamical representation in the sense of the Dynamical Systems Theory. Nevertheless it is possible to prove, for inter-temporal models, the existence of an equilibrium manifold, and in this manifold to describe,"equilibrium paths" and to show the possible future behavior of the economy. This representation doesn't follows from endogenous dynamic laws, it is predetermined by the endowments as functions of the time.

The existence of singular economies, i.e., economies with endowments for which zero is a singular value of the excess utility function, implies the possibility of the existence of abrupt

[24]the positive cone in L_∞, has not empty interior.

changes along an equilibrium path, even for a not singular economy. So the possibility of foreseen is absolutely lost, see [Accinelli, E., Puchet, M.].

How to characterize an equilibrium by intrinsic dynamical properties, and how to give sense to the concept of *evolution* are open challenges for Equilibrium Theory.

As in the Plato's statement may be that a totally satisfactory solutions to these questions be unreachable with the current theory, but our uncertainty could be, at least partially, assuaged by economics and mathematics.

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