Separability in Stochastic Binary Systems

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***Abstract*—A Stochastic Binary System (SBS) is a mathematical model of multi-component on-off systems subject to random failures. SBS models extend classical network reliability models (where the components subject to failure are nodes or links of a graph) and are able to represent more complex interactions between the states of the individual components and the operation of the system under study.**

The reliability evaluation of stochastic binary systems belongs to the class of $\mathcal{NP}$-Hard computational problems. Furthermore, the number of states is exponential with respect to the size of the system (measured in the number of components). As a consequence, the representation of an SBS becomes a key element in order to develop exact and/or approximation methods for reliability evaluation.

The contributions of this paper are three-fold. First, we present the concept of *separable* stochastic binary systems, showing key properties, such as an efficient representation and complexity in the reliability evaluation. Second, we fully characterize separable systems in two ways, using a geometrical interpretation and minimum-cost operational subsystems. Finally, we show the application of separable systems in network reliability models, specifically in the all-terminal reliability model, which has a wide spectrum of applications.

***Index Terms*—Stochastic Binary System, Network Reliability, Computational Complexity, Chernoff Inequality.**

I. Introduction

In system reliability analysis, the goal is to find the probability of correct operation of a system subject to component failures. A common practical problem is to design a system with maximum reliability meeting budget constraints [1], [2], [3], [4].

Classical network reliability analysis shaped the body of this field. In this basic setting, we are given a connected graph G with perfect nodes, and the links work independently with identical probability r. The *all-terminal reliability*, $R_G(r)$, is the probability that the resulting subgraph remains connected. This model and some variants (such as perfect links and nodes subject to failure) has been employed to model reliability of classical communications networks, where the emphasis was on a fixed infrastructure of sites holding communication equipment and of fixed links connecting them. Nevertheless, these models have limitations to represent the more diverse landscape of communication networks infrastructure, relying on different equipment, paradigms, and particularly in the case of wireless networks, where usually there does not exist a fixed, predetermined topology. Stochastic binary systems (SBS) generalize the static reliability concept to any system composed of a number of components subject to independent failures with known probabilities, and where the operation or failure of the system as a whole is a function of the state of the individual components. In this sense, SBS are a more flexible tool for evaluating and optimizing the reliability of a wider spectrum of real systems, both in the networking area and in other quite different applications area [5], [6], [7], [8]. At the same time, SBS present their own challenges in terms of computational analysis, as the evaluation of the reliability a general stochastic binary system belongs to the class of $\mathcal{NP}$-Hard problems. This has motivated different research efforts, tackling efficient exact methods for some subclasses of SBS, as well as approximations for the general case [9], [10], [11], [12].

In this paper, we propose a novel representation of a special subset of stochastic binary systems, called *separable systems*. This representation is explored in order to better understand its benefits and potential application in network reliability analysis. The contributions of this paper can be summarized in the following items:

- An efficient representation of separable systems is proposed. It considers $N+1$ real numbers, being N the size of the system (measured as the number of components subject to failure).
- A full geometrical characterization of separable systems is introduced. Furthermore, a second characterization considers minimum-cost operational subsystems and minimum-capacity cutsets. Interestingly enough, these characterizations provide a *bridge* between functional analysis and system reliability.
- The concept of separability in graphs is introduced.
- The all-terminal reliability evaluation of separable graphs is investigated.

This paper is organized as follows. Section II presents fundamental concepts of stochastic binary systems. Separable systems are introduced in Section III. They can be represented more efficiently using $N+1$ real numbers instead of 2^N numbers that are used to represent arbitrary SBS. Two different characterizations of separable systems are proposed in Section IV. A particular analysis of the all-terminal reliability model is offered in Section V. Finally, Section VII has concluding remarks and trends for future work.

II. Stochastic Binary Systems

The following terminology is adapted from [13].

Definition 1 (Stochastic Binary System)**.** A *stochastic binary system* is a triad (S, r, ϕ):

- $S = \{1, \dots, N\}$ is a ground set of *components*,
- $r = (r_1, \dots, r_N)$ are their *elementary reliabilities*, and
- $\phi : \{0,1\}^N \to \{0,1\}$ is the *structure*.

The concept of reliability is generalized to arbitrary stochastic binary systems.

Definition 2 (Reliability/Unreliability)**.** Let $\mathcal{S} = (S, p, \phi)$ be a stochastic binary system, and consider a random vector $X = (X_1, \dots, X_N)$ with independent coordinates governed by Bernoulli random variables such that $P(X_i = 1) = r_i$. The *reliability* of $\mathcal{S}$ is the probability of correct operation of the system:

$$R_{\mathcal{S}} = P(\phi(X) = 1) = E(\phi(X)) = \sum_{x:\phi(x)=1} P(X = x). \quad (1)$$

The *unreliability* of $\mathcal{S}$ is $U_{\mathcal{S}} = 1 - R_{\mathcal{S}}$.

A stochastic binary system is *homogeneous* if the elementary reliabilities are identical (i.e., $r_i = r$ for all i). In this paper we deal with homogeneous SBS.

Definition 3 (Pathsets/Cutsets)**.** Let $\mathcal{S} = (S, r, \phi)$ be a stochastic binary system. A possible state or configuration $x \in \{0,1\}^N$ is a *pathset* (resp. *cutset*) if $\phi(x) = 1$ (resp., if $\phi(x) = 0$).

The binary set $\{0,1\}$ is equipped with the partial order, defined by $0 \leq 0$, $0 \leq 1$ and $1 \leq 1$. The set $\{0,1\}^N$ inherits a natural order in the Cartesian product. Given two partially ordered sets A and B, a function $f : A \to B$ is monotonically increasing if $f(a_1) \leq f(a_2)$ whenever $a_1 \leq a_2$. As usual, we denote $y < x$ if $y \leq x$ and $y \neq x$. Let us denote by $\overline{0}_N$ (resp. $\overline{1}_N$) the binary word with all bits set to 0 (resp. to 1), and by δ_i the binary word with all bits in 0 except the bit in position i which is set to 1.

Definition 4 (Stochastic Monotone Binary System (SMBS))**.** The triad $\mathcal{S} = (S, r, \phi)$ is a *stochastic monotone binary system* if the structure function $\phi : \{0,1\}^N \to \{0,1\}$ is monotonically increasing, $\phi(\overline{0}_N) = 0$ and $\phi(\overline{1}_N) = 1$.

Observe that SMBS represent *well-behaved* SBS, in the sense that, given a working configuration, the system can fail after the removal of some components, but can not fail if some failed components start to work. Additionally, the system does not work if it has no operational components, and the full-system works.

Definition 5 (Minpaths/Mincuts/Rays)**.** Let $\mathcal{S} = (S, r, \phi)$ be an SMBS:

- A pathset x is a *minpath* if $\phi(y) = 0$ for all $y < x$.
- A cutset y is a *mincut* if $\phi(x) = 1$ for all $x > y$.
- The x-ray is the set $S_x = \{y \in \{0,1\}^N : y \geq x\}$.

It is worth to remark that an SMBS is fully characterized by its mincuts (or its minpaths). In fact, if we are given the complete list of minpaths, then the complete list of pathsets is precisely the union of the x-rays for some minpath x.

We will denote by $\overline{x}$ the state complementary to x in bits (i.e., 0 in x are set to 1 in $\overline{x}$, and vice-versa). In particular, $\overline{\phi(x)} = 1 - \phi(x)$. The following definition of duality will be useful for our later analysis of monotonicity and bounds [14]:

Definition 6 (Duality)**.** The dual of a stochastic binary system $\mathcal{S} = (S, r, \phi)$ has identical ground set S, elementary reliabilities $r_i^d = 1 - r_i$, and structure $\phi^d(x) = 1 - \phi(\overline{x})$, for all possible states $x \in \{0,1\}^N$. The dual is denoted by $\mathcal{S}^d = (S, 1 - r, \phi^d)$.

The following examples provide an insight of the different applications of stochastic binary systems. Classical examples include a reference in the field for the interested reader.

1) All-Terminal Reliability: the ground set is precisely the links of a simple graph. The system is up if the resulting random graph is connected.
2) K-Terminal Reliability: in the same random graph, the system is up if some distinguished node-set K, called terminals, belong to the same connected component [15].
3) Diameter Constrained Reliability: a diameter constraint d is added to the K-Terminal Reliability. The system is up if every pair of terminals are connected by paths whose length is not greater than the diameter [16], [17].
4) Node-Reliability: the ground set is the set of the nodes of a simple graph. The system is up if the resulting random graph is connected.
5) Node-Edge Reliability: both links and nodes fail in a random graph. The system is up if and only if the resulting subgraph is connected [18].
6) k-N-Survivability: the system is up if and only if there are at least k identical components in operational state out of N. This homogeneous system is also known as k-out-of-m system. We will denote $\phi_{(k,N)}$ to its structure [10].
7) k-N-Degraded: the system is down if and only if there are at least k identical components in failure state out of N. We will denote $\psi_{(k,N)}$ to its structure. Clearly: $\psi_{(k,N)} = \phi_{(N-k,N)}$.
8) Feasibility: consider an arbitrary integer linear program P a set of constraints $Ax \leq b$, for instance coming from a integer linear program P, with binary decision variables $x_1, \dots, x_N$. If the x are not arbitrary but instead correspond to events such that the elementary reliability r_i is the likelihood of the event $x_i = 1$, or $p_i = 1/2$ if there is no available experiment, and the structure is $\phi(x) = 1$ if x is feasible for P, then the reliability is the probability that the random vector x meets the constraints.

There exists an interplay between SBS and propositional logic. Recall that a theorem-proving procedure is the first $\mathcal{NP}$-Complete decision problem established by Stephen Cook [19]. In other words, the recognition of a tautology is a hard decision problem from propositional logic.

Theorem 1. *The reliability evaluation of an arbitrary SMBS belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. Arnie Rosenthal formally proved that the reliability evaluation for the K-terminal reliability model belongs to the class of $\mathcal{NP}$-Hard computational problems [20]. Since K-Terminal is a particular SMBS, the result follows by inclusion. □

Corollary. *The reliability evaluation of an arbitrary SBS belongs to the class of $\mathcal{NP}$-Hard problems.*

Theorem 2. *The determination of a cutset in an arbitrary SBS is an $\mathcal{NP}$-Complete decision problem.*

Proof. Consider an arbitrary propositional logic φ with m literals. Build the corresponding SBS with m elements and φ as the structure. Then, φ is a tautology if and only if the corresponding SBS has no cutsets. □

The following result is in strong contrast with Theorem 2. It is useful to build pointwise reliability estimation in SMBS [10]:

Proposition 1. *A mincut can be found using m rule-evaluations in an arbitrary SMBS.*

Proof. The evidence is Algorithm 1. Clearly, it requires m evaluations, and returns a cutset x. We will prove the statement in two steps:

(1) The state x is also a mincut.
(2) Algorithm 1 is optimal in terms of rule-evaluations.

For (1), suppose that the output x is not a mincut. Therefore, there exists some $j \in \{1, \ldots, m\}$ such that $\phi(x + e_j) = 0$. Let us denote $x^{(j)}$ the state for the iteration j in the **for**-loop. Observe that x is (possibly) increased in each iteration. Therefore, $x^{(j)} \leq x \leq x + e_j$. Since $\phi(x + e_j) = 0$ and ϕ is monotonous, we get that $\phi(x^{(j)}) = 0$. But in this case the j-esime bit would have been set to 1, and this bit is set to 0 in the output. This is impossible, since the bits in x are only increased during the execution of Algorithm 1.
For (2), observe that in the worst case the null vector $x = 0$ is a mincut. The only way to determine that 0 is a mincut is the test $\phi(e_i) = 1$ for all possible canonical vectors $\{e_i\}_{i=1,\ldots,m}$, and it requires m rule-evaluations. □

Algorithm 1 $x = Mincut(m, \phi)$

$x \leftarrow 0$
for $i = 1$ **to** m **do**
  $y \leftarrow x + e_i$
  if $\phi(y) = 0$ **then**
    $x \leftarrow y$
  end if
end for
return x

Proposition 1 is constructive. It provides an interplay with propositional logic. Nevertheless, the algorithmic complexity of finding a mincut in arbitrary SMBS is still an open problem. In other words, we do not know yet if the quadratic algorithm presented in Proposition 1 is optimal.

Les us close this section with three elementary properties of the dual system that will be useful in our analysis. Now, we additionally consider the elementary reliabilities represented by the vector r. In specific context of network reliability analysis, the determination of the cutset with maximum probability is useful. Let us study the problem for arbitrary SMBS. First, three elementary results [14]:

Lemma 1. *The dual of the dual is the original system.*

Proof. $\phi^{d^d}(x) = 1 - \phi^d(\overline{x}) = 1 - (1 - \phi(x)) = \phi(x)$. □

Lemma 2. *The dual of an SMBS is another SMBS.*

Proof. Consider arbitrary states $x \leq y$ and a monotone structure ϕ. Since $\overline{x} \geq \overline{y}$, we get that $\phi(\overline{y}) \leq \phi(\overline{x})$. Therefore: $\phi^d(x) = 1 - \phi(\overline{x}) \leq 1 - \phi(\overline{y}) = \phi^d(y)$. □

Since if the original system is homogeneous, the dual system is homogeneous as well, and we get the following result:

Corollary. *Consider a homogeneous SMBS. Then, a state x is a pathset with maximum probability if and only if $\overline{x}$ is a cutset with maximum probability in the dual.*

Proof. First, assume that x is a pathset with maximum probability. Then $\phi(x) = 1$, and $\phi^d(\overline{x}) = 1 - \phi(x) = 0$, so $\overline{x}$ is a cutset in the dual system. Assume that x has precisely h elements in operational state. Then $P(x) = r^h(1-r)^{m-h}$. In the dual the elementary reliability equals $1 - r$. Then, in the dual the probability is $P(\overline{x}) = (1-r)^{m-h} r^h$, identical to the probability of x in the original system. The converse holds by Lemma 1.

□

From Corollary II, we can study pathsets instead of cutsets, and the results under monotonicity hold. Recall that we want to find a cutset with maximum probability in an SMBS.

Proposition 2. *The determination of a pathset with maximum probability in arbitrary SMBS belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. Consider the K-Terminal Reliability model, in the homogeneous case. Since the model is homogeneous, a pathset with maximum probability is precisely a minimum cardinality minpath. But in the K-Terminal model, this is the Steiner Problem in Graphs, which belongs to the $\mathcal{NP}$-Hard class [21].

□

Corollary. *The determination of a cutset with maximum probability in arbitrary SMBSs belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. Combine Proposition 2 with Corollary II and Lemma 2. (Proposition) □

Corollary II is a negative result that has a deep impact in the understanding of SMBS. In the homogeneous Source-Terminal Reliability model, finding a mincut with maximum probability is precisely the problem of finding a minimum cardinality s-t cutset. Using the theory of flows in networks, it is known that the cardinal of such a mincut is the maximum flow between s and t with unit capacities in the links. Therefore, Corollary II discards any possibility of the existence of finding in SMBS efficient algorithms similar to those corresponding to maximum flow theory, unless $\mathcal{P} = \mathcal{NP}$.

There is however a particular family of SBS where the whole reliability polynomial can be obtained in polynomial-time. In fact, if the number of pathsets is a polynomial $P(N)$ in the size N, the reliability can be computed in polynomial time as the probability sum among all pathsets.

Definition 7 (Weak SBS)**.** An infinite sequence of SBS $\mathcal{S}_n = (S_n, p_n, \phi_n)$ is a *weak-sequence* if the number of pathsets is a polynomial in the number of components.

The reliability polynomial can be fully obtained in weak-sequences in polynomial time.

Tue dual system has complementary reliability with respect to the original one:

Lemma 3. *If $\mathcal{S} = (S, \phi, p)$ is an SBS, then $R_{\mathcal{S}^d} = 1 - R_{\mathcal{S}}$.*

Proof. Recall that the dual system has complementary probabilities in every component. Therefore: $P^d(X = x) = \prod_{i:x_i=1}(1 - r_i)\prod_{i:x_i=1}(r_i) = P(X = \overline{x})$. Let $\mathcal{P}$ denote the path-sets of the original SBS. Then:

$$\begin{aligned} R_{\mathcal{S}^d} &= \sum_{x:\phi^d(x)=1} P^d(X = x) = \sum_{x:\phi(\overline{x})=0} P^d(X = x) \\ &= 1 - \sum_{x:\phi(\overline{x})=1} P^d(X = x) \\ &= 1 - \sum_{x:\phi(\overline{x})=1} P(X = \overline{x}) \\ &= 1 - P(x \in \mathcal{P}) = 1 - R_{\mathcal{S}}. \end{aligned}$$

□

III. Separable Systems

Observe that $\{0,1\}^N$ is the set of the extremal points of the unit hypercube $Q_N \subseteq \mathbb{R}^N$. Let us assign labels to the extremal points of Q_N according to a given structure ϕ. Every hyperplane defines a partition of $\mathbb{R}^N$ into two subsets. Consider the family of hyperplanes $\mathcal{H}$ such that $\overline{0}_N$ and $\overline{1}_N$ lie on different sides. For any member H of $\mathcal{H}$, denote by $Q_0 \subseteq Q_N$ the extremal points of the hypercube that belong to the side of $\overline{0}_N$; and $Q_1 = Q_N - Q_0$. Define a structure function ϕ_H such that its cutsets are precisely Q_0, and its pathsets are Q_1. Consider an equivalence relation $(\mathcal{H}, \sim)$ such that $H_1 \sim H_2$ if and only if $\phi_{H_1} = \phi_{H_2}$.

Recall that in the Euclidean space $\mathbb{R}^N$, a hyperplane is fully characterized by a normal vector $\vec{n}$ and a point P that belongs to the hyperplane: $\langle \vec{n}, X - P\rangle = 0$, where $\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$ is the inner product. If we denote $\vec{n} = (n_1, \ldots, n_N)$ and $\langle \vec{n}, P\rangle = \alpha_0$, the hyperplane can be written as $\sum_{i=1}^{N} n_i x_i = \alpha_0$.

Lemma 4. *If $\phi = \phi_H$ for some hyperplane H, then there exists $H_2 \sim H_1$ with non-negative normal vector such that $\|\vec{n}\|_1 = \sum_{i=1}^{N} n_i = 1$.*

Proof. Let $\phi = \phi_H$ for the hyperplane $H) \sum_{i=1}^{N} n_i x_i = \alpha_0$, and suppose that there exists some index j such that $n_j < 0$. There are two exhaustive and mutually disjoint cases:

i There exists some mincut $x = (x_1, \ldots, x_N)$ such that $x_j = 0$: in this case, we know that $x + \delta_j$ is a minpath, so, $\phi(x+\delta_j) = 1$. By the definition of the hyperplane, we get that $\sum_{i=1}^{N} n_i x_i \leq \alpha_0$ but $\sum_{i=1}^{N} n_i x_i + n_j > \alpha_0$. The only possibility is that $n_j > 0$. But we assumed $n_j < 0$; this is a contradiction.

ii All mincuts verify $x_j = 1$: Consider an alternative hyperplane $H_2) \sum_{i\neq j}^{N} n_i x_i = \alpha_0 - n_j$. We will prove that $H_2 \sim H$. If x is a mincut, then $\sum_{i=1}^{N} n_i x_i \leq \alpha_0$, and therefore $\sum_{i\neq j}^{N} n_i x_i \leq \alpha_0 - n_j$. If x is a minpath, it must have $x_j = 1$. Since $\sum_{i=1}^{N} n_i x_i > \alpha_0$ we get that $\sum_{i\neq j}^{N} n_i x_i > \alpha_0 - n_j$. Observe that $n_j = 0$ in the new hyperplane H_2, and $H_2 \sim H$ as desired.

By an iterative replacement of all the negative coordinates we obtain an equivalent hyperplane $H_2 \sim H$ with non-negative vector $\vec{n'}$, expressed by $H_2) \sum_{i=1}^{N} n_i' x_i = \alpha'$ for some real number α'. Finally, observe that $\overline{0}_N$ is always a cutset, so $0 \leq \alpha'$. Analogously, $\overline{1}_N$ is always a pathset, so $\sum_{i=1}^{m} n_i' > \alpha' \geq 0$. The result is obtained by a normalization of the normal vector $\vec{n_2}$, which is possible since $\sum_{i=1}^{N} n_i' > 0$. □

Even though there exist infinite equivalent hyperplanes, using Support Vector Machine (SVM) it is possible to find a single hyperplane with the largest gap (this is, with the largest distance to any of the vertices in the hypercube). Using Lemma 4, we can replace it by an equivalent hyperplane with non-negative versor. Without loss of generality, we will assume a non-negative normal vector with unit 1-norm.

Proposition 3. *The structures ϕ_H are monotone.*

Proof. By Lemma 4, in particular we can choose $n_i \geq 0$ in the hyperplane $H) \sum_{i=1}^{N} n_i x_i = a_0$. Let us denote $f(x) = \sum_{i=1}^{N} n_i x_i$. If $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$, and therefore $\phi_H(x_1) \leq \phi_H(x_2)$. □

A subtlety is that the *mincuts* from Lemma 4 are indeed the points $Q_0 \subset Q_N$ that are closer to the original hyperplane. A natural question is to determine if all SMBS can be represented by a hyperplane. The answer is negative:

Proposition 4. *There exist SMBS that cannot be represented by a hyperplane.*

Proof. Consider the SMBS defined by the mincuts $M = \{(1,1,0,0),(0,0,1,1)\}$. Observe that the set of states $P = \{(0,1,0,1),(1,0,1,0)\}$ is a subset of minpaths. Suppose for a moment that there exists some separator $H) \sum_{i=1}^{4} n_i x_i = \alpha$ for some real numbers $\alpha, n_1, \ldots, n_4$. Since $(1,1,0,0)$ and $(0,0,1,1)$ are mincuts, we get that $\sum_{i=1}^{4} n_i \leq 2\alpha$. However, $(1,1,0,0)$ and $(0,0,1,1)$ are minpaths, so $\sum_{i=1}^{4} n_i > 2\alpha$; a contradiction. □

Definition 8 (Separable System)**.** An SBS is *separable* if the cutsets/pathsets can be separated by some hyperplane.

An interpretation of separable systems recalls Riesz representation theorem for Hilbert spaces [22]. Indeed, the structure of a separable system can be written as an indicator that an inner-product exceeds some threshold in a Hilbert space:

$$\phi(x) = 1_{\langle x, \vec{n} \rangle \geq \alpha_0}. \tag{2}$$

Even though separable systems accept an efficient representation, the reliability evaluation is hard in nature:

Proposition 5. *The reliability evaluation of separable systems belong to the class of $\mathcal{NP}$-Hard problems.*

Proof. By reduction from $PARTITION$. Consider an instance of natural numbers $A = \{a_1, \ldots, a_N\}$, and let $S = \sum_{i=1}^{N}$ be the sum over the elements of the list. Let us define $\alpha_i = \frac{a_i}{S}$, $\alpha_{min} = \min_{i=1,\ldots,N}\{\alpha_i\}$, and consider the separable systems $\mathcal{S}_1$ and $\mathcal{S}_2$:

1) The separable system $\mathcal{S}_1$ characterized by the hyperplane $\sum_{i=1}^{N} \alpha_i x_i = \frac{1}{2} + \frac{\alpha_{min}}{2}$;
2) The separable system $\mathcal{S}_2$ characterized by the hyperplane $\sum_{i=1}^{N} \alpha_i x_i = \frac{1}{2}$;

Observe that the difference of the reliability of both systems evaluated at $p = 1/2$ is:

$$\begin{aligned} & R_{\mathcal{S}_2}(1/2) - R_{\mathcal{S}_1}(1/2) \\ &= P(\sum_{i=1}^{N} \alpha_i X_i \geq \frac{1}{2}) - P(\sum_{i=1}^{N} \alpha_i X_i \geq \frac{1}{2} + \frac{\alpha_{min}}{2}) \\ &= P(\sum_{i=1}^{N} \alpha_i X_i = \frac{1}{2}) \\ &= \frac{\#\{(x_1, \ldots, x_N) \in \{0,1\}^N : \sum_{i=1}^{N} \alpha_i x_i = \frac{1}{2}\}}{2^N}, \end{aligned}$$

and the last number is positive if and only if there exists a subset $B \subseteq \{1, \ldots, N\}$ such that $\sum_{i \in B} \alpha_i = \frac{1}{2}$. In that case, if we multiply on both sides by S we get that $\sum_{i \in B} a_i = \frac{S}{2}$, and the answer to PARTITION for the list A is YES. Otherwise, the answer to PARTITION is NO. Therefore, the reliability evaluation of separable systems is at least as hard as PARTITION, and it belongs to the class of $\mathcal{NP}$-Hard problems. □

The reader can appreciate that Proposition III is a generalization of Theorem 1. In the following, we build reliability bounds for separable systems, and as a corollary we find bound for arbitrary SBS.

IV. Characterization of Separable Systems

A natural question is to characterize separable systems in terms of pathsets and cutsets. Let us denote $CH(\mathcal{P})$ and $CH(\mathcal{C})$ the convex hull of the pathsets and cutsets respectively.

Theorem 3. *An SBS is separable iff $CH(\mathcal{P}) \cap CH(\mathcal{C}) = \emptyset$.*

Proof. If the intersection is empty, Hahn-Banach separation theorem for convex sets asserts that there exists a hyperplane H that separates those convex sets [22]. As a consequence, $\phi = \phi_H$ for some hyperplane H.

For the converse, we know that the SBS is separable. Therefore, there exists some hyperplane $H = \sum_{i=1}^{N} n_i x_i = \alpha_0$ such that $\sum_{i=1}^{N} n_i x_i \leq \alpha_0$ for cutsets, and $\sum_{i=1}^{N} n_i x_i > \alpha_0$ for pathsets. Suppose for a moment that $CH(\mathcal{P}) \cap CH(\mathcal{C}) \neq \emptyset$. There exists some element $z \in \mathbb{R}^N$ such that:

$$z = \sum_{j=1}^{r} \alpha_j x_j = \sum_{k=1}^{s} \beta_k y_k, \tag{3}$$

for some states $x_1, \ldots, x_r \in P$, $y_1, \ldots, y_s \in C$, and non-negative numbers such that $\sum_{j=1}^{r} \alpha_j = \sum_{k=1}^{s} \beta_k = 1$. If we denote $x_j = (x_{j1}, \ldots, x_{jN})$ we know that $\sum_{i=1}^{N} n_i x_{ji} > \alpha_0$. Therefore, for $z = (z_1, \ldots, z_N)$ we get that:

$$\begin{aligned} \sum_{i=1}^{N} n_i z_i &= \sum_{i=1}^{N} n_i (\sum_{j=1}^{r} \alpha_j x_{ji}) \\ &= \sum_{j=1}^{r} \alpha_j [\sum_{i=1}^{N} n_i x_{ji}] \\ &> (\sum_{j=1}^{r} \alpha_j) \alpha_0 = \alpha_0. \end{aligned}$$

Analogously, using the fact that $z = \sum_{k=1}^{s} \beta_k y_k$ we get that $\sum_{i=1}^{N} n_i z_i \leq \alpha_0$, which is a contradiction. Therefore we must have $CH(\mathcal{P}) \cap CH(\mathcal{C}) = \emptyset$, and the result holds. □

By Proposition 3 we have a full geometrical characterization of separable systems, which accept an efficient representation.

In the following, we consider an alternative characterization, in terms of weighted cutsets and pathsets. Consider an arbitrary assignment $\alpha_1, \ldots, \alpha_N$ of non-negative numbers to the respective components of the system. The condition $\sum_{i=1}^{N} \alpha_i x_i \geq \alpha_0$ for all the pathsets is equivalent to find the pathset x with minimum-cost, $c(x) = \sum_{i:x_i=1} \alpha_i$, and test if $c(x) \geq \alpha_0$. Analogously, the condition $\sum_{i=1}^{N} \alpha_i y_i < \alpha_0$ for all the cutsets is equivalent to test whether the cutset y with minimum-cost, $c(y) = \sum_{i:y_i=0} \alpha_i$, satisfies the test $S - c(y) < \alpha_0$, being $S = \sum_{i=1}^{N} \alpha_i$ the cost of the global system. Observe that, for convenience, the cost of a cutset is defined as the sum of the components under failure. In particular, we get the following characterization of separable systems:

Theorem 4. *An SBS is separable if and only if there exists an assignment of non-negative costs to the components $\{\alpha_i\}_{i=1,\ldots,N}$ such that $S < c(y) + c(x)$, being $c(x)$ and $c(y)$ the pathset/cutset with minimum-cost respectively.*

Proof. First, let us assume that we have a separable SBS with hyperplane $\sum_{i=1}^{N} \alpha_i x_i = \alpha_0$. Using the previous reasoning, the assignment $\{\alpha_i\}_{i=1,\ldots,N}$ verifies $c(x) \geq \alpha_0$ and $S > c(y) + \alpha_0$. Therefore, $S > c(y) + c(x)$.
For the converse, let us fix $\alpha_0 = c(x)$, the pathset with minimum cost. Clearly, the specific pathset x meets the condition $\sum_{i=1}^{N} \alpha_i x_i \geq \alpha_0$; in fact the equality is met. By its definition, the inequality holds for the other pathsets, since they have greater cost. Finally, we use the fact that $S < c(y) + c(x)$ to verify that the cutset with minium-cost, y, meets the inequality $\sum_{i=1}^{N} \alpha_i y_i < \alpha_0$. The inequality for the other cutsets is straight since y is a cutset with minimum-cost. Therefore, the SBS is separable, concluding the proof. □

V. Separability in Graphs

Our characterization of separable systems has a straight reading in the celebrated all-terminal reliability model.

Definition 9 (Separable Graph)**.** A graph $G = (V, E)$ is separable if there exists an assignment of non-negative real numbers $\alpha_1, \ldots, \alpha_m$ of its m links, and there exists a threshold α such that $c(E') \geq \alpha$ if and only if the spanning subgraph $G' = (V, E')$ is connected.

Let G be a connected graph. Recall that Kruskal algorithm provides efficiently the cost of the minimum spanning tree, $MST(G)$. Furthermore, the cutset with minimum-cost, $m(G)$, is obtained using Ford-Fulkerson algorithm. Therefore, the following corollary of Theorem 4 holds for graphs:

Corollary. *A graph is separable iff there exists an assignment $\{\alpha_i\}_{i=1,\ldots,N}$ to the links such that $S < MST(G) + m(G)$, being $MST(G)$ the cost of the minimum spanning tree, and $m(G)$, the mincut with minimum capacity.*

Intuitively, if the graph is dense enough, it is hard to exceed the global cost S of the graphs using the minimum spanning tree and mincut. Our first result deals with complete graphs:

Proposition 6. *A complete graph K_n is never separable for any $n \geq 4$*

Proof. Consider an arbitrary assignment $\{\alpha_i\}_{i=1,\ldots,n(n-1)/2}$ to the links of K_n, and an arbitrary star-graph $K_{1,n}$ contained in K_n. Since $K_{1,n}$ is connected, its cost is greater than, or equal to the minimum spanning tree, so, $c(K_{1,n}) \geq MST(K_n)$. Furthermore, the complementary links of $K_{1,n}$, or the complementary graph $K_{1,n}^C$, is a cutset (it isolated a single node), so the cost must exceed the mincut: $c(K_{1,n}^C) \geq m(K_n)$. But then, the global cost is $c(K_n) = c(K_{1,n}) + c(K_{1,n}^C) \geq MST(K_n) + m(K_n)$. The conclusion is that $c(K_n) \geq MST(K_n) + m(K_n)$ for any feasible assignment, and K_n is not separable. □

Proposition 7. *Given a graph $G(V, E)$, we define a graph $G'(V, E')$ with $E' = E \cup \{e\}$. if G is not separable then G' either.*

Proof. For the converse we suppose that G' is separable. Therefore, there exists an assignment of non-negative real numbers $\{\alpha_i\}_{i=1,\ldots,n+1}$ of its links such that $\sum_{i=1}^{n} \alpha_i + \alpha_{n+1} \leq MST(G') + m(G')$. For Ford-Fulkerson we know that $m(G') \leq m(G) + \alpha_{n+1}$. By definition we have that $MST(G') \leq MST(G)$. Therefore $\sum_{i=1}^{n} \alpha_i + \alpha_{n+1} \leq MST(G') + m(G') \leq MST(G) + m(G) + \alpha_{n+1}$ thus $\sum_{i=1}^{n} \alpha_i + \cancel{\alpha_{n+1}} \leq MST(G) + m(G) + \cancel{\alpha_{n+1}}$. Since G is not separable we get a contradiction. □

Proposition 8. *Given a graph $G(V, E)$, we define a graph $G'(V', E')$ with $V' = V \cup \{\hat{v}\}$, $E' = E \cup \{(\hat{v}, x)\}$, $x \in V$ and $\hat{v} \notin V$. if G is not separable then G' either.*

Proof. For the converse we suppose that G' is separable. Therefore, there exists an assignment of non-negative real numbers $\{\alpha_i\}_{i=1,\ldots,N+1}$ ($|V| = N$ and $|V'| = N+1$) of its links such that $\sum_{i=1}^{n+1} \alpha_i \leq MST(G') + m(G')$. Without losing generality we assume that α_{N+1} is the cost associated to $(\hat{v}, x)$. It is clear that $(\hat{v}, x) \in MST(G')$ so $\sum_{i=1}^{N} \alpha_i + \cancel{\alpha_{N+1}} \leq MST(G) + \cancel{\alpha_{N+1}} + m(G')$,
$\sum_{i=1}^{N} \alpha_i \leq MST(G) + m(G)$ where $m(G') \leq m(G)$ is used in the last inequality. Since G is not separable we get a contradiction. □

Proposition 9. *Given a graph $G(V, E)$, we define a graph $G'(V', E')$ with $V' = V \cup \{\hat{v}\}$, $E' = E \cup \hat{E}$, $\hat{v} \notin V$ and $\hat{E} = \{(\hat{v}, x) | x \in V\}$. if G is not separable then G' either.*

Proof. Mathematical induction in $|E'|$.
Base case: $|E| = 1$ holds by Proposition 8.
Inductive step:
Assume that $|E'| = p$ holds. Consider: $|E'| = p + 1$
Let G' be a graph with $|E'| = p + 1$ with E' set of links from $\hat{v}$ ($\hat{v} \notin G$) to G. Let denote $\hat{e}$ a link belonging to E'. Consider $\hat{E} = E' - \{\hat{e}\}$ and $\hat{G} = G'(V', E' - \{\hat{e}\})$. We know

by inductive hypothesis that $\hat{G}$ is not separable. By proposition 7 we know that $\hat{G} \cup \{\hat{e}\}$ (link $\hat{e}$ has both vertices in $\hat{G}$) is not separable. □

Proposition 10. *Given a graph $G(V,E)$ and a path graph $H(\bar{V},\bar{E}) = \{(x,y_1),(y_1,y_2),\ldots,(y_m,z)\}$ with $V \cap \bar{V} = \{x,z\}$, we define a graph $G'(V',E')$ with $V' = V \cup \bar{V}$, $E' = E \cup \bar{E}$. if G is not separable then G' either.*

Proof. Let G_1 be a graph that $G_1 = G \cup \{(x,y_1)\}$. By proposition 8 G_1 is not separable. Let G_2 be a graph that $G_2 = G_1 \cup \{(y_1,y_2)\}$. By proposition 8 G_2 is not separable. Extending our reasoning let G_m be a graph that $G_m = G_{m-1} \cup \{(y_{m-1},y_m)\}$. By proposition 8 G_m is not separable. Let G_{m+1} be a graph that $G_{m+1} = G_m \cup \{(y_m,z)\}$. Clearly $G_{m+1} = G'$. We know that G_m is not separable. Given that $y_m \in G_m$ and $z \notin G_m$ by proposition 8, $G_m \cup \{(y_m,z)\}$ is not separable. Therefore $G' = G \cup H$ is not separable. □

Proposition 11. *All graphs excluding cycle graphs and 2-connected graph (2-vertex-connected graphs and 2-edge-connected) are not separable.*

Proof. Let G be a 2-connected graph that it is not a cycle graph. Given that G is 2-connected, by Frederickson-Ja'Ja' theorem there exists a decomposition such that $G = H_0 \cup H_1 \cup H_2 \cup \cdots \cup H_m$ with H_0 is a cycle and $\{H_i\}_{i\in\{1,\ldots,m\}}$ are H-paths over the previous graph. Let $\hat{G}$ be a graph that $\hat{G} = H_0 \cup H_1$. We know that $\hat{G}$ is a Monma graph therefore it is not separable. Let us denote by G_i a graph such that $G_i = \hat{G} \cup H_2 \cup \cdots \cup H_i$, with $i \in 2\ldots m$. By proposition 10 we know that G_i, with $i \in 2\ldots m$ are not separable. They are H-path aggregations over a non separable graph. By way of $G_m = G$, G is not separable. □

Proposition 12. *The Halim graphs are not separable.*

Proof. Let H be a Halim graph. H can be written as $H = T \cup C$ where T is a tree where all internal vertices have degree greater than 2 and C is a cycle graph that pass by all tree leaves in order. T is embebed in C. There exists a path graph $P \subset T$ such that $C \cup P$ is a Monma graph. We denote $G = C \cup P$. $\hat{G}$ is not separable. H is obtained from $\hat{G}$ adding links with one or both extremal points in the graph under construction. In any of the 2 ways, each aggregation results in a non-separable graph. Therefore H is not separable. □

Proposition 13. *Given two cycle graphs G_1 and G_2 with $G_1 \cap G_2 = \emptyset$, let us define $\hat{G} = G_1 \cup \hat{e} \cup G_2$ as the concatenation of both cycles and a bridge edge ($\hat{e}$) between them. $\hat{G}$ is separable.*

Proof. Question: There exists an assignment of non-negative numbers: $\{\alpha_i\}_{i\in 1,\ldots,m_1}$ to the links of G_1, with $|G_1| = m_1$, $\{\beta_i\}_{i\in 1,\ldots,m_2}$ to the links of G_2, with $|G_2| = m_2$, and α to $\hat{e}$ such that: $\sum_{i=1}^{m_1}\alpha_i + \sum_{i=1}^{m_2}\beta_i + \alpha \le MST(\hat{G}) + m(\hat{G})$. If the answer is yes, $\hat{G}$ is separable. Without losing generality we can assume that: $\alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots \le \alpha_{m_1}$ and $\beta_1 \le \beta_2 \le \beta_3 \le \cdots \le \beta_{m_2}$. We know that $m(\hat{G}) = min\{\alpha_1 + \alpha_2, \beta_1 + \beta_2, \alpha\}$, $MST(\hat{G}) = MST(G_1) + MST(G_2) + \alpha$, $MST(G_1) = \sum_{i=1}^{m_1-1}\alpha_i$ and $MST(G_2) = \sum_{i=1}^{m_2-1}\beta_i$. We have that

$$\sum_{i=1}^{m_1}\alpha_i + \sum_{i=1}^{m_2}\beta_i \le \sum_{i=1}^{m_1-1}\alpha_i + \sum_{i=1}^{m_2-1}\beta_i + m(\hat{G}) \qquad (4)$$

We know that $\alpha_{m_1} \ge \alpha_2 \ge \alpha_1$ and $\beta_{m_2} \ge \beta_2 \ge \beta_1$. Let's take $\alpha_{m_1} = \beta_{m_2} = \alpha_2 = \beta_2 = \alpha_1 = \beta_1 = \hat{\alpha}$, thus $m(\hat{G}) = min\{\hat{\alpha}+\hat{\alpha}, \hat{\alpha}+\hat{\alpha}, 2\hat{\alpha}\} = 2\hat{\alpha}$. We have that $m(\hat{G}) = 2\hat{\alpha}$. Replacing this result in equation 4 we have that

$$\sum_{i=1}^{m_1-1}\alpha_i + \cancel{\hat{\alpha}} + \sum_{i=1}^{m_2-1}\beta_i + \cancel{\hat{\alpha}} \le \sum_{i=1}^{m_1-1}\alpha_i + \sum_{i=1}^{m_2-1}\beta_i + \cancel{2\hat{\alpha}}$$

Taking $\alpha_i = \hat{\alpha}\ \forall i \in 1,\ldots,m_1$ $\beta_i = \hat{\beta}$, $\forall i \in 1,\ldots,m_2$ and $\alpha = 2\hat{\alpha}$, with $\hat{\alpha} > 0$ fixed, the separability of $\hat{G}$ holds. □

VI. Proof-of-Concept

Our goal is to understand the performance of our bounding method for some sample situations. We consider a wireless system subject to node failures. For this reason we consider the all-terminal Node-Reliability model. Recall that Node-Reliability is not an SMBS in general.

We considered the graphs sketched in Figures 1-4. For each graph we consider the SBS given by Node Reliability with structure ϕ. Then, we find the closest SMBS ϕ_m and ϕ^u, the closest separator systems $\underline{\phi}$, $\overline{\phi}$ and ϕ^*, solving the respective ILP formulations and CPLEX optimization engine. Tables I and II report the misclassification errors and reliability bounds respectively, using $r_i = r = 1/2$ for the elementary reliabilities. The asterisk $*$ means that the optimization reached the limit of three hours, and this value is sub-optimal; this is the case of the Icosahedron graph (I).

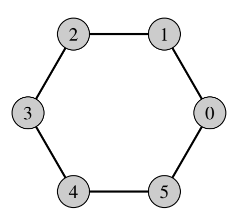

Fig. 1. Elementary cycle C_6

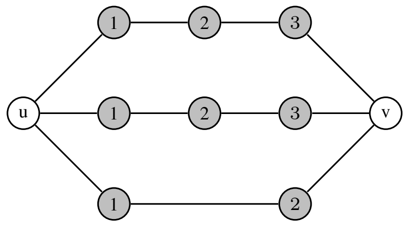

Fig. 2. Monma graph $M_{(3,3,2)}$.

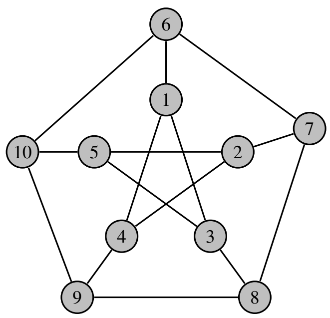

Fig. 3. Petersen graph (P)

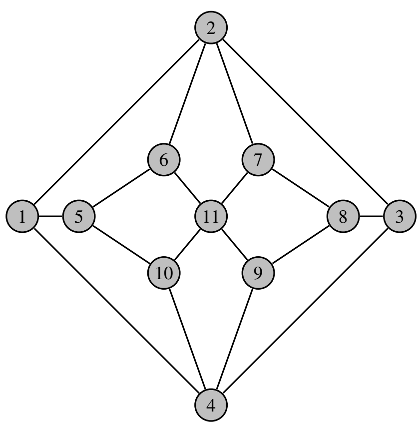

Fig. 4. Icosahedron graph (I)

TABLE I
MISCLASSIFICATION ERROR

Case	$d(\phi,\phi_m)$	$d(\phi,\phi^u)$	$d(\phi,\underline{\phi})$	$d(\phi,\overline{\phi})$	$d(\phi,\phi^*)$
C_6	12	21	12	23	12
M_{10}	106	720	113	759	113
P	175	390	314	425	190
I	302	958	537	958	379*

TABLE II
PERFORMANCE OF RELIABILITY AND BOUNDS

Case	$R_{\mathcal{S}}(\phi)$	$R_{\mathcal{S}}(\phi^*)$	$R_{\mathcal{S}}(\underline{\phi})$	$R_{\mathcal{S}}(\overline{\phi})$	LB	UB
C_6	0.3906	0.2031	0.2031	0.7500	0	1
M_{10}	0.1846	0.0801	0.0742	0.9258	0	1
P	0.5449	0.5059	0.2383	0.9600	0	1
I	0.5317	0.4932	0.2695	0.9995	0	1

Observe that ϕ^* achieves the minimum distance, $d(\phi,\phi_m) \leq d(\phi,\underline{\phi})$, and $d(\phi,\phi^u) \leq d(\phi,\overline{\phi})$, as expected by definition. However, the gaps are small. This suggests that finding bounds for an arbitrary SBS by separable systems (which are advantageous due to their small space requirement for representation) may not entail a large loss of precision when compared by bounds obtained using SMBS approximations (which potentially need exponential space for representation).

From Table II we can check that $R_{\mathcal{S}}(\underline{\phi}) \leq R_{\mathcal{S}}(\phi) \leq R_{\mathcal{S}}(\overline{\phi})$, while $R_{\mathcal{S}}(\phi^*)$ is closer to $R_{\mathcal{S}}(\phi)$. The last two columns UB is the upper bound found applying Lemma **??** directly and using the separator hyperplane for $\overline{\phi}$. Column LB is calculated applying Theorem **??** and using the separator hyperplane for $\underline{\phi^d}$. For the four cases under study, these formulations result in trivial bounds.

Figures 5-8 display the exact reliabilities $R_{\mathcal{S}}(\underline{\phi}) \leq R_{\mathcal{S}}(\phi) \leq R_{\mathcal{S}}(\overline{\phi})$, together with Chernoff bounds, for each network. The non-monotonicity of $R_{\mathcal{S}}(\phi)$ is appreciated Icosahedron network topology. The bounds provided by the separable systems are much tighter than the Chernoff bounds. At the same time, there is a clear gap between lower and upper bounds, specially for medium-range values of p. When p approaches 1, the quality of the bounds improve; and particularly the lower bound is closer to the exact value. This is of interest, as usually when designing or evaluating a system, the goal is to guarantee a certain level of reliability (thus, making the lower bound a relevant approximation).

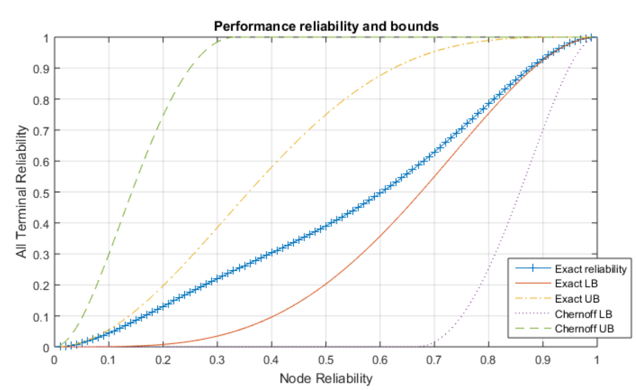

Fig. 5. C6 graph

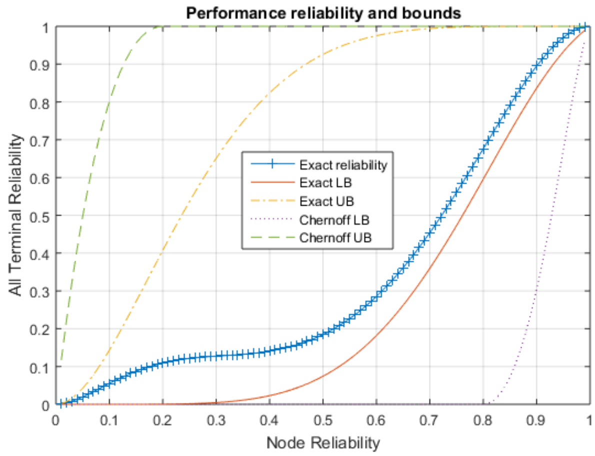

Fig. 6. M10 graph

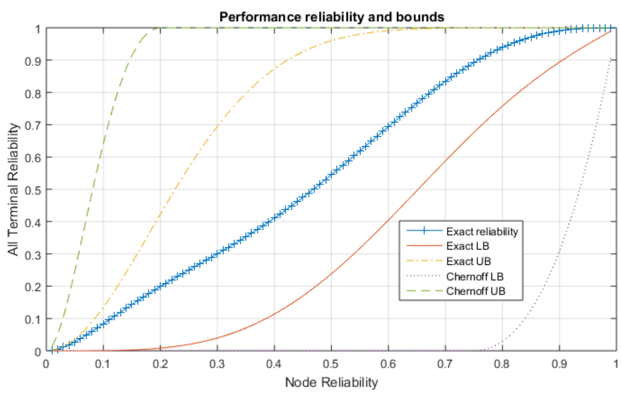

Fig. 7. Petersen graph

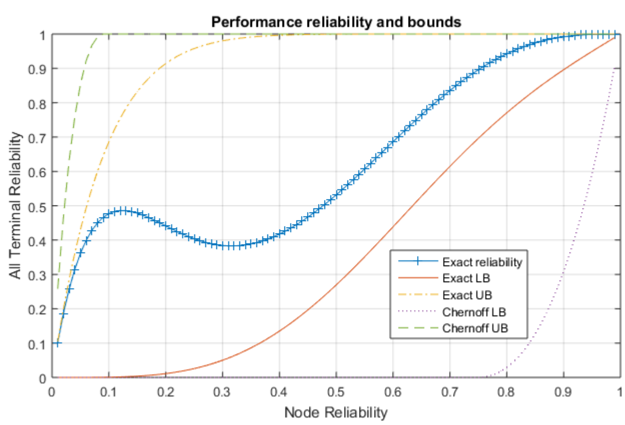

Fig. 8. Icosahedron graph

VII. Concluding Remarks

An efficient representation of separable systems is here introduced. This representation is analogous to Riesz Representation Theorem for Hilbert spaces, but for particular SBS, using a simple inner product. Supported by this natural representation, we produce reliability bounds for arbitrary SBS, exploiting duality and Chernoff inequality. The results are highlighted in systems under the node-reliability model.

This interesting interplay between Stochastic Binary Systems and Functional Analysis should be further studied. As a future work, we would like to develop new reliability bounds using the theory of Functional Analysis, and apply these results to potential applications in real-life systems. Other lines of research include taking into account dependencies between the components' states, and studying how SBS structure can be exploited in a dynamic context (i.e, when the time dimension is taken into account so that the components' states are evolving, i.e, failing and being repaired, at different moments of the system evolution).

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References

[1] Y. Chopra, B. Sohi, R. Tiwari, and K. Aggarwal, "Nework topology for maximizing the terminal reliability in a computer communication network," *Microelectronics Reliability*, vol. 24, no. 5, pp. 911 – 913, 1984.

[2] H. W. Lee, K. Lee, and E. Modiano, "Maximizing reliability in wdm networks through lightpath routing," *IEEE/ACM Transactions on Networking*, vol. 22, no. 4, pp. 1052–1066, Aug 2014.

[3] J. Barrera, H. Cancela, and E. Moreno, "Topological optimization of reliable networks under dependent failures," *Operations Research Letters*, vol. 43, no. 2, pp. 132 – 136, 2015.

[4] D. Coudert, J. Luedtke, E. Moreno, and K. Priftis, "Computing and maximizing the exact reliability of wireless backhaul networks," in *International Network Optimization Conference (INOC)*, Lisbon, Portugal, 2017.

[5] J. Barta and Montemanni, "Wireless multicasting under probabilistic node failures: a heuristic approach," *Optimization and Engineering*, vol. 13, no. 4, pp. 705–726, 2012.

[6] F. Markowetz and M. Boutros, *Linking Phenotypes and Genotypes*, ser. Cambridge Series in Systems Genetics. Cambridge University Press, 2015.

[7] H. Cancela, G. Guerberoff, F. Robledo, and P. G. Romero, "Reliability maximization in stochastic binary systems," in *14th International Conference on the Design of Reliable Communication Networks (DRCN 2018)*, Paris, France, February 2018.

[8] X. Song, Z. Zhai, Y. Liu, and J. Han, "A stochastic approach for the reliability evaluation of multi-state systems with dependent components," *Reliability Engineering & System Safety*, vol. 170, no. Supplement C, pp. 257 – 266, 2018.

[9] M. Lin, "An efficient algorithm for computing the reliability of stochastic binary systems," *IEICE Transactions*, vol. 87-D, no. 3, pp. 745–750, 2004.

[10] E. Canale, H. Cancela, J. Piccini, F. Robledo, P. Romero, G. Rubino, and P. Sartor, "Recursive variance reduction method in stochastic monotone binary systems," in *2015 7th International Workshop on Reliable Networks Design and Modeling (RNDM)*, Oct 2015, pp. 135–141.

[11] E. Canale, H. Cancela, F. Robledo, P. Romero, and P. Sartor, "Diameter constrained reliability: Complexity, distinguished topologies and asymptotic behavior," *Networks*, vol. 66, no. 4, pp. 296–305, 2015.

[12] G. Levitin, *Computational Intelligence in Reliability Engineering: Evolutionary Techniques in Reliability Analysis and Optimization*, ser. Studies in Computational Intelligence. Springer Berlin Heidelberg, 2006.

[13] M. O. Ball, "Computational complexity of network reliability analysis: An overview," *IEEE Transactions on Reliability*, vol. 35, no. 3, pp. 230 –239, aug. 1986.

[14] P. Romero, "Duality in stochastic binary systems," in *2016 8th International Workshop on Resilient Networks Design and Modeling (RNDM)*, Sept 2016, pp. 85–91.

[15] M. O. Ball, “Complexity of network reliability computations,” *Networks*, vol. 10, no. 2, pp. 153–165, 1980.

[16] L. Petingi and J. Rodriguez, “Reliability of networks with delay constraints,” in *Congressus Numerantium*, vol. 152, 2001, pp. 117–123.

[17] E. Canale, H. Cancela, F. Robledo, P. Romero, and P. Sartor, “Full complexity analysis of the diameter-constrained reliability,” *International Transactions in Operational Research*, vol. 22, no. 5, pp. 811–821, 2015.

[18] R. Dash, N. Barpanda, P. Tripathy, and C. Tripathy, “Network reliability optimization problem of interconnection network under node-edge failure model,” *Applied Soft Computing*, vol. 12, no. 8, pp. 2322 – 2328, 2012.

[19] S. A. Cook, “The complexity of theorem-proving procedures,” in *Proceedings of the third annual ACM symposium on Theory of computing*, ser. STOC ’71. New York, NY, USA: ACM, 1971, pp. 151–158.

[20] A. Rosenthal, “Computing the reliability of complex networks,” *SIAM Journal on Applied Mathematics*, vol. 32, no. 2, pp. 384–393, 1977.

[21] R. M. Karp, “Reducibility among combinatorial problems,” in *Complexity of Computer Computations*, R. E. Miller and J. W. Thatcher, Eds. Plenum Press, 1972, pp. 85–103.

[22] J. Conway, *A Course in Functional Analysis*, ser. Graduate Texts in Mathematics. Springer New York, 1994.